THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
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The Mathematical Association.

THE ANNUAL MEETING will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, on *Monday, 6th January*, 1930, at 2.30 p.m., and *Tuesday, 7th January*, 1930, at 10.0 a.m. and 2.30 p.m.

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THE GENERALISATION OF GEOMETRICAL OPTICS.*

BY A. BUXTON, M.A.

GEOMETRICAL Optics as a part of the mathematical curriculum in schools, universities, and technical colleges is now a dead subject,† yet the generalised study of ray-tracing in optics started by Sir William Rowan Hamilton in 1824 is well worth attention on all sides. He writes: "Those who have meditated on the beauty and utility in theoretical mechanics of the general method of Lagrange, who have felt the power and dignity of that central dynamical theorem which he deduced in the *Mécanique Analytique* must feel that mathematical optics can only then attain a coordinate rank with mathematical mechanics when it shall possess an appropriate method and become the unfolding of a central idea." With this end in view, Sir Wm. Hamilton submits his three papers on the Theory of Systems of Rays to the Royal Irish Academy, which, as Lord Rayleigh said, bear the mark of genius throughout. The enormous task that Sir Wm. Hamilton set himself will be evident when we discuss the wanderings of rays through optical systems in an elementary way to-night leading up to Hamilton's general method of the Characteristic Function and its development.

I. We will first, then, give a brief résumé of the methods of tracing a ray of light through a symmetrical optical system consisting of a series of spherical

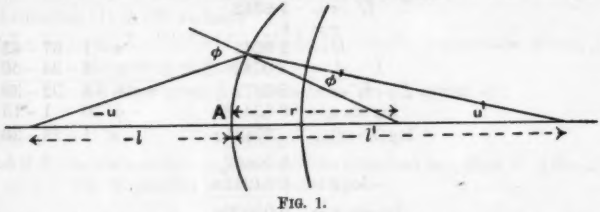

FIG. 1.

surfaces symmetrically placed with regard to the central axis. They are (*a*) trigonometrical, (*b*) algebraic, (*c*) graphical.

(*a*) *Let us take the case of a ray intersecting the axis,* then the course of the ray through each surface is plainly given by the formulae:

$$\frac{-l-r}{\sin\phi} = -\frac{r}{\sin u}, \quad \text{giving } \phi \qquad \mu\sin\phi = \mu'\sin\phi', \quad \text{giving } \phi';$$

$$\phi + u = \phi' + u', \quad \text{giving } u'; \qquad \frac{r}{\sin u'} = \frac{l'-r}{\sin\phi'}, \quad \text{giving } l'.$$

* Paper delivered to the Cardiff Mathematical Society Oct. 22/23.
† Prof. Filon, *Trans. Opt. Soc.* 22, 1921 (208).

If the next surface is distant d, $\dfrac{l'-d-r}{\sin\phi'}$=etc., and so on, making the necessary allowances for signs, the ray can be traced through each surface in turn and therefore through the system. To the computer there is a real difficulty as regards signs, but this disappears if one convention is adhered to throughout.

One such arrangement is as follows:

(1) All lengths l, r are counted +ve to the right, −ve to the left, of the vertex A of the surface under consideration.

(2) The acute angle u, u' negative if counter-clockwise, and *vice versa*.

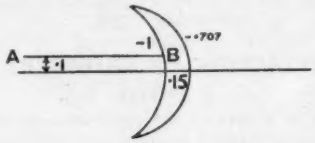

FIG. 2.

Hence the scheme for AB parallel to the axis and at height ·1 unit above it is as follows:

$\dfrac{\gamma}{r}=\dfrac{\cdot1}{-1}=\sin\phi$

$\log\gamma$	9·0000	u 0
$-\log r$	·0000n	ϕ − 5° 44′ 21″
$\log\sin\phi$	9·0000n	$u+\phi$ − 5° 44′ 21″
$-\log\mu'$	− ·18127	$-\phi'$ + 3° 46′ 38″
$+\log\sin\phi'=$	8·81873n	u' − 1° 57′ 43″
$\log r=$	·0000n	
	8·81873	
$-\log\sin u'=$	−8·53448n	
$\log(L'-r)=$	·28425n	
$L'-r=$	−1·9242	
$r=$	−1	
$L'=$	−2·9242	u − 1 − 57 − 43
$L'-d_1=$	−3·0742	ϕ − 6 − 34 − 56
$L'-d_1-r=$	−2·3672	$u+\phi$ − 8 − 32 − 39
$\log\sin u=$	8·53448n	$-\phi'$ 10 − 1 −15
$\log(L-r)$	·37424n	u' 1 − 28 − 36
	8·90872	
$-\log r=$	9·84942n	
$\log\sin\phi=$	9·05930n	
$+\log N=$	·18127	
$\log\sin\phi'=$	9·24057n	
$\log r$	9·84942n	
	9·08999	
$-\log\sin u'$	−8·41111	
$\log(L'-r)=$	·67888	
$L'-r=$	4·7740	
$+r=$	− ·707	
$L'=$	4·0670	

(*b*) When the ray does not intersect the axis it is said to be skew; the tracing of a skew ray is not so simple, but an approximate scheme was mapped out by Gauss and more accurately by Steinheil. Time will not permit us to go fully into their proofs this evening, and so we pass on to the second type of method which incidentally solves the "skew" ray problem as well, but which is less suitable for numerical computation.

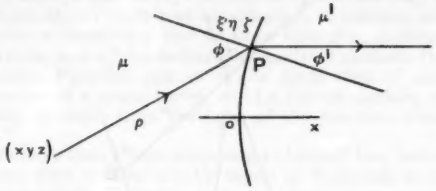

FIG. 3.

Let the incident ray be

$$\frac{X-\xi}{L}=\frac{Y-\eta}{M}=\frac{Z-\zeta}{N}=\rho,$$

then the coordinates of P where the incident ray meets the first surface are

$$\left.\begin{aligned}\xi&=X-L\rho\\ \eta&=Y-M\rho\\ \zeta&=Z-N\rho\end{aligned}\right\} \tag{1}$$

The equation of the surface taking the origin O where the symmetrical axis cuts the first surface, and axis of symmetry as the axis of x is

$$\frac{2x}{R}=x^2+y^2+z^2, \tag{2}$$

where R is the curvature, and therefore the direction cosines of the normal at (ξ, η, ζ) are $1-\xi R,\ -\eta R,\ -\zeta R$.

$$\cos\phi=L(1-\xi R)-\eta RM-\zeta RN, \tag{3}$$

$$\cos\phi'=L'(1-\xi R)-\eta RM'-\zeta RN'. \tag{4}$$

Substituting (1) in (2) we have

$2(X-L\rho)=R\{(X-L\rho)^2+(Y-M\rho)^2+(Z-N\rho)^2\}$ a quadratic giving ρ;

(1) now give ξ, η, ζ,

(3) then gives ϕ; $\mu\sin\phi=\mu'\sin\phi'$, giving ϕ';

$$\frac{\mu'}{\sin\phi}=\frac{\mu}{\sin\phi'},$$

so that if KA is the incident ray and AC the refracted ray, then if $AB=\mu$ and $AC=\mu'$, BC will be parallel to the normal at A.

Project on to the x-axis:

$$\frac{\mu'L'-\mu L}{1-\xi R}=-\mu\cos\phi+\mu'\cos\phi', \text{ etc.},$$

giving finally L', M', N', and therefore defining the final ray.

The latter method is clearly not suitable for computation unless modified.

(*c*) The third method, viz. the graphical weapon, is singularly unfortunate in the majority of these problems, for in practice the angles are generally small, and a small error in drawing means a big displacement of the image point, consequently the work has to be carried out generally by calculation. Enough has been said to show that much cumbersome numerical work is necessary in tracing a ray through a system of a large number of surfaces.

For example, Fraunhofer constructed a system consisting of four surfaces. Bessel investigated the lens but did not compute any skew rays, and later Steinheil computed seven rays which met the axis and nine which did not

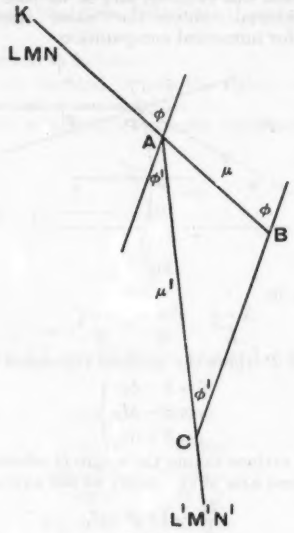

FIG. 4.

(see diagram). Prof. Sampson estimates that for each ray which meets the axis there are fully fifty operations and more than 200 for each skew ray. Steinheil calculated the rays as a matter of fact which impinge at 48′ to the axis at the points given, *i.e.* for one inclination to the axis there are for four surfaces 2150 operations.

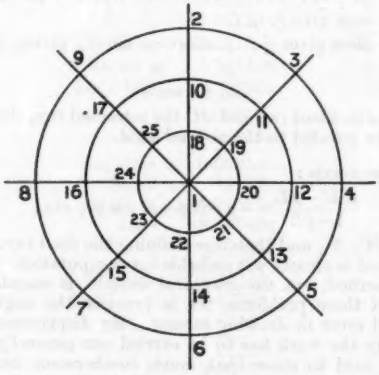

FIG. 5.

Now as mathematicians are notorious for their dislike of numerical calculations, much work was done in order to avoid it. There were two magnificent attempts to generalise the problem, one due to Gauss (1840), and the other due to Hamilton (1824). If we are in the least doubtful of Hamilton's boldness in attacking the general problem, let us listen to what he has to say—"the theory of Systems of Rays is an attempt to establish general principles respecting the systems of rays produced by the ordinary reflexion (or refraction) of light at any mirror or combination of mirrors, shaped and placed in any manner whatsoever; and to show that the mathematical properties of such a system may all be deduced by analytic methods from the form of one Characteristic Function just as in the application of analysis to geometry, the properties of a plane curve, or of a curved surface, may all be deduced by uniform methods from the form of the function which characterises its equation."

Here we have then a man proposing to himself long before the above special calculations were worked out by Bessel or Steinheil, to trace out by means of a Characteristic Function *any* ray, straight or curved [not necessarily at 48′ to the axis], through any system not *necessarily* symmetrical with surfaces of any curvatures not necessarily spheres in any succession of media homogeneous or heterogeneous. The magnitude of the task was appalling, but despite this fact his success was so marked that little has been developed by others since which cannot be found in a more generalised form in his work.

II. *The generalisations.*

The work of Gauss in 1840 defined an optical system by certain reference planes having special properties.*

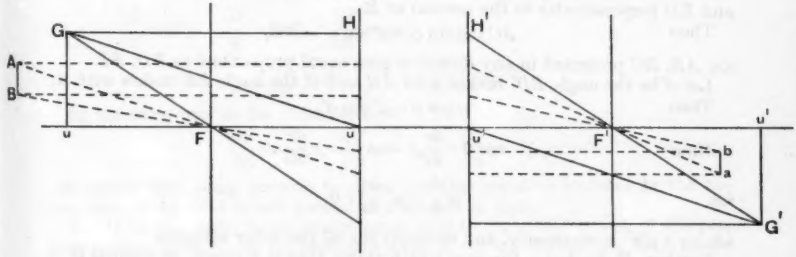

FIG. 6.

For example, *GH* passing through the axial point at infinity in the object space will pass through *F′*, the second principal focal on the image side, and the plane through *F′* perpendicular to the axis of symmetry is a principal focal plane. Then there is a principal focal plane through *F* on the object side. The ray through *F′* will intersect the second unit plane in *H′* and the first unit plane such that $HU = H'U'$, the planes then for which the magnification of the system is unity. Knowing the positions of these planes the image could be constructed graphically from the given object without any of the inaccuracies due to the ordinary graphical methods notwithstanding the violent distortion of distances perpendicular to the axis in comparison with those parallel to the axis. Homographic and symbolical methods were given for finding the positions of these planes for any optical system. The assumption that the object and image surfaces are planes normal to the axis of symmetry of the system is not borne out in practice, and Clerk Maxwell

* *A system of lenses*, Charles Pendlebury.

indicated that the whole conception is inconsistent with the known properties of light.

There are two theorems upon which the method of Hamilton rests: (*a*) Fermat's theorem, (*b*) Malus' theorem. Fermat's theorem is known as the Principle of Least Time in which if r is the length of the ray in medium μ, $\Sigma\mu r$ *the* **reduced path** from the first to the final point is stationary, *i.e.* the adjacent path differs from it by quantities of the second order. In other words, $\Sigma\mu r$ is the same for small displacements of points where the ray meets the surfaces.

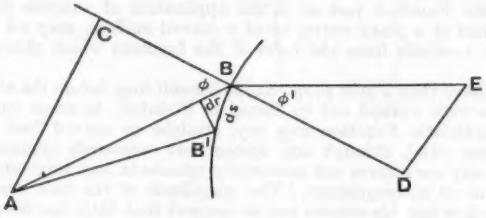

FIG. 7.

For consider a single surface:

Let AB be the incident ray, BB' *the element, ds an arbitrary displacement from B.*

Let AB and BD represent to same scale μ and μ' respectively. Draw AC and ED perpendicular to the normal at B.

Then $$AC=\mu\sin\phi=\mu'\sin\phi'=DE,$$

i.e. AB, BC projected in any direction give equal projections as BD, BE.

Let θ be the angle BB' makes with AB and θ' the angle BE makes with it.

Then $$\mu\cos\theta=\mu'\cos\theta'.$$

Now $$\cos\theta=\frac{dr}{ds},\quad \cos\theta'=-\frac{dr'}{ds},$$

i.e. $$\mu\frac{dr}{ds}+\mu'\frac{dr'}{ds}=0,$$

i.e. $\mu r+\mu' r'$ is stationary, and similarly for all the other surfaces.

Further, Malus has a theorem which states that if a pencil be normal to a surface before refraction the refracted pencil will also be normal to some surface.

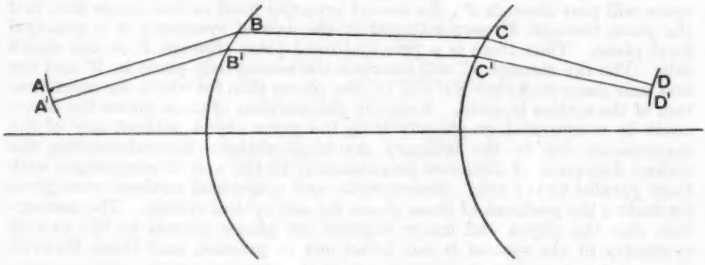

FIG. 8.

For $\Sigma\mu r$ for $ABCD = \Sigma\mu r$ for $A'B'C'D'$, then $ABCD$ by Fermat's theorem is equal to $AB'C'D$.

Since $A'B'$ is normal to a surface at A',

$$A'B' = AB',$$

i.e. $$AB'C'D = A'B'C'D = A'B'C'D',$$

i.e. $C'D = C'D'$, *i.e.* DD' is orthogonal to the final direction of the rays.

Rays of a pencil emanating from (x, y, z) are orthogonal to a sphere, and so after refraction will be orthogonal to some surface.

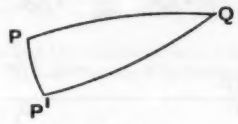

FIG. 9.

Let $$V = \Sigma\mu r.$$

This reduced path, called the Characteristic Function, is a measure of the time from $x'y'z'$ to xyz and is a function of these coordinates.

Let P be the point x', y', z' and Q the point

$$(x'+dx', \quad y'+dy', \quad z'+dz').$$

Let the orthogonal surface through P cut the consecutive ray through Q in P', then

$$\begin{aligned} dV &= \mu P'Q = \mu(\text{projection of } PQ \text{ on consecutive ray}) \\ &= \mu PQ\cos\theta = \mu[(l'+dl')dx' + (m'+dm')dy' + (n'+dn')dz', \end{aligned}$$

i.e. $$\frac{\partial V}{\partial x'} = \mu l', \quad \frac{\partial V}{\partial y'} = \mu m', \quad \frac{\partial V}{\partial z'} = \mu n',$$

or $$\left(\frac{\partial V}{\partial x'}\right)^2 + \left(\frac{\partial V}{\partial y'}\right)^2 + \left(\frac{\partial V}{\partial z'}\right)^2 = \mu'^2.$$

In the same way in the object space

$$\frac{\partial V}{\partial x} = -\mu_0 l, \quad \frac{\partial V}{\partial y} = -\mu_0 m, \quad \frac{\partial V}{\partial z} = -\mu_0 n,$$

the minus sign being present to allow for the negative increase in the coordinates from right to left instead of from left to right.

Let O be the origin in the image space, and let P, Q be the feet of the perpendiculars from O on to emergent rays l, m, n; $l+dl, m+dm, n+dn$ respectively.

Let P be x', y', z' and Q, $x'+dx', y'+dy', z'+dz'$, then we have the two relations

$$lx' + my' + nz' = 0,$$

$$(l+dl)(x'+dx') + (m+dm)(y'+dy') + (n+dn)(z'+dz') = 0,$$

giving $$x'\,dl + y'\,dm + z'\,dn = -(l\,dx' + m\,dy' + n\,dz'),$$

i.e. $$dV = -\mu(x'\,dl + y'\,dm + z'\,dn), \qquad n = \sqrt{1-l^2-m^2},$$

$$dV = -\mu\left(x'\,dl + y'\,dm - \frac{z'l}{n}dl - \frac{z'm}{n}dn\right), \quad dn = -\frac{(l\,dl + m\,dm)}{n}$$

$$= -\mu\left\{dl\left(x' - \frac{z'l}{n}\right) + dm\left(y' - \frac{z'm}{n}\right)\right\},$$

or $$\frac{\partial V}{\partial l} = -\mu\left(x' - \frac{z'l}{n}\right)$$

$$\frac{\partial V}{\partial m} = -\mu\left(y' - \frac{z'm}{n}\right).$$

The emergent ray being $\frac{X-x'}{l}=\frac{Y-y'}{m}=\frac{Z-z'}{n}=\rho$, we have that if ξ, η, ζ are the coordinates of the point where the emergent ray meets the $Z=0$ plane, then

$$\frac{\partial V}{\partial l}=-\mu\xi, \quad \frac{\partial V}{\partial m}=-\mu\eta.$$

Similarly in the object space we have

$$\frac{\partial V}{\partial l_0}=+\mu_0 x, \quad \frac{\partial V}{\partial m_0}=+\mu_0 y.$$

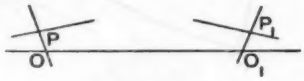

FIG. 9A.

Hence V may be regarded as a function of l_0, m_0, l, m, where V is now the time to pass from P to P_1, and since there is symmetry about the axis it may therefore be expanded in powers of $l_0^2+m_0^2$, ll_0+mm_0, l^2+m^2, which we may call p, q, r.

$$\frac{\partial V}{\partial l}=2l\frac{\partial V}{\partial r}+l_0\frac{\partial V}{\partial q}=-\mu\xi,$$
$$\frac{\partial V}{\partial m}=2m\frac{\partial V}{\partial r}+m_0\frac{\partial V}{\partial q}=-\mu\eta.$$

Now any point on the emergent ray is

$$\frac{X-\xi}{l}=\frac{Y-\eta}{m}=\frac{Z}{n}=\rho,$$

i.e.

$$X=\xi+l\rho,$$
$$Y=\eta+m\rho,$$
$$Z=n\rho,$$

and along with these we have

$$\xi+\frac{2l}{\mu}\frac{\partial V}{\partial r}=-\frac{l_0}{\mu}\frac{\partial V}{\partial q},$$
$$\eta+\frac{2m}{\mu}\frac{\partial V}{\partial r}=-\frac{m_0}{\mu}\frac{\partial V}{\partial q},$$

i.e. if

$$\rho=\frac{2}{\mu}\frac{\partial V}{\partial r}, \quad \frac{X}{Y}=\frac{l_0}{m_0},$$

which means that the point (X, Y) lies in the plane containing the axis of symmetry and the line with direction l_0, m_0, n_0.

In other words, the principal focus in the image space for the direction $l_0m_0n_0$ in the object space is distant $\frac{2}{\mu}\frac{\partial V}{\partial r}$ past where the ray meets $Z=0$ in the image space.

Hence, since the principal focal distance is measured from the second unit plane to the principal focus, we have the coordinates of the principal focus as

$$l_0f, \ m_0f,$$

i.e.

$$l_0f=-\frac{l_0}{\mu}\frac{\partial V}{\partial q},$$
$$m_0f=-\frac{m_0}{\mu}\frac{\partial V}{\partial q},$$

or
$$\frac{1}{\mu}\frac{\partial V}{\partial q} = -f',$$

and similarly for the object space

$$\frac{1}{\mu_0}\frac{\partial V}{\partial q} = -f.$$

We thus have here a sort of potential function in optics which on being differentiated gives the secondary focal length, just as in theoretical mechanics the components of the forces can be deduced by differentiation from the potential function.

Let us illustrate by finding this characteristic function called by Bromwich the Modified Function and by Bruns the Eikonal for a single spherical surface.

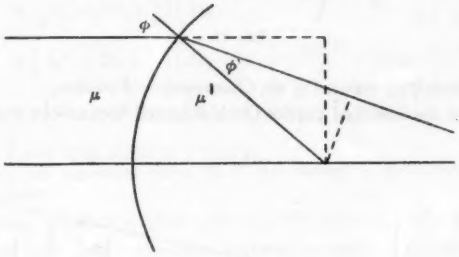

FIG. 10.

$$\begin{aligned}
V &= -\mu r\cos\phi + \mu' r\cos\phi' \\
&= r(\mu'\cos\phi' - \mu\cos\phi), \\
&= r\sqrt{\mu'^2\cos^2\phi' + \mu^2\cos^2\phi - 2\mu\mu'\cos\phi'\cos\phi} \\
&= r\sqrt{\mu'^2 + \mu^2 - (\mu'\sin\phi' - \mu\sin\phi)^2 - 2\mu\mu'\cos(\phi - \phi')} \\
&= r\sqrt{\mu'^2 + \mu^2 - 2\mu\mu'(ll_0 + mm_0 + nn_0)} \\
&= r\sqrt{\mu'^2 + \mu^2 - 2\mu\mu'(ll_0 + mm_0 + \sqrt{1-l^2-m^2}\sqrt{1-l_0^2-m_0^2}} \\
&= r\sqrt{\mu'^2 + \mu^2 - 2\mu\mu'(q + \sqrt{1-p} + \sqrt{1-r}),}
\end{aligned}$$

$$\frac{\partial V}{\partial q} = -\frac{\mu\mu' r}{(\mu'\cos\phi' - \mu\cos\phi)}, \quad \text{i.e. } f' = \frac{\mu' r}{(\mu'\cos\phi' - \mu\cos\phi)}.$$

The old method checks this result as follows:

$$\frac{CF}{\sin\phi'} = \frac{PF}{\sin\phi} = \frac{r}{\sin(\phi - \phi')} = \frac{r}{\cos\phi'\sin\phi - \sin\phi'\cos\phi},$$

i.e.
$$\frac{\mu' CF}{\mu'\sin\phi'} = \frac{\mu PF}{\mu\sin\phi} = \frac{\mu\mu' r}{\mu\mu'\sin\phi\cos\phi' - \mu\mu'\sin\phi'\cos\phi}$$

$$= \frac{\mu\mu' r}{\mu'\sin\phi'(\mu'\cos\phi' - \mu\cos\phi)},$$

i.e.
$$\mu' CF = \mu PF = \frac{\mu\mu' r}{\mu'\cos\phi' - \mu\cos\phi},$$

$$CF = \frac{\mu' r}{\mu'\cos\phi - \mu\cos\phi} \quad \text{as before.}$$

For a number of surfaces the problem is more complicated; different origins have to be taken and the parts of V have to be expanded in powers of p, q, r and the usual approximations made.

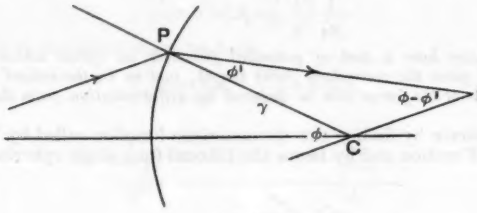

FIG. 11.

III. *Clerk Maxwell's treatment of the Characteristic Function.*

In vol. ii. of his collected papers Clerk Maxwell tackles the same problem.

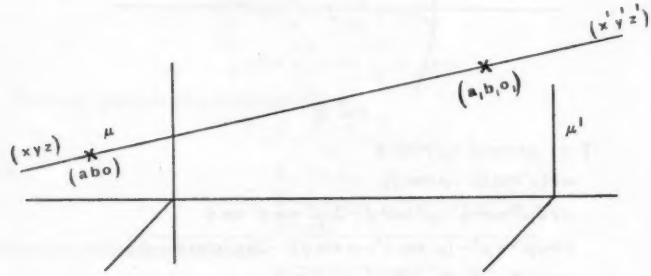

FIG. 12.

His function, however, is

$$V = \mu\{(x-a)^2+(y-b)^2+z^2\}^{\frac{1}{2}} + U + \mu'\{(x'-a_1)^2+(y'-b_1)^2+z'^2\}^{\frac{1}{2}},$$

where U is the reduced path from the point (a, b) in the $z=0$ plane in the object space to the point (a_1b_1) in the $z=0$ plane in the image space.

He then uses the fact that the function is stationary for values of a, b, a_1, b, having previously proved that since by Fermat's theorem U the reduced path from (a, b, c) to $(a_1b_1c_1)$ is stationary for all points of incidence of the ray on the successive surfaces, then U' the reduced path from (x, y, z) to $x'y'z'$ along the same ray [*i.e.* through the system *via* (a, b, c) and the same points of incidence and out *via* $(a_1b_1c_1)$ to $x'y'z'$] is stationary for any arbitrary variations of (a, b, c) and $(a_1b_1c_1)$, and then for simplicity c and c_1 are taken as zero.

Now U is therefore a function of a, b, a_1, b_1, and in the symmetrical system these four variables may be reduced to three, viz. a^2+b^2, $a_1^2+b_1^2$, aa_1+bb_1.

Hence

$$V = \mu\left\{z + \frac{(a-x)^2+(b-y)^2}{2z}\right\} + k_0 + k_1(a^2+b^2) + k_2(a_1^2+b_1^2) + k_3(aa_1+bb_1)$$
$$+ \mu'\left\{z' + \frac{(a_1-x')^2+(b_1-y')^2}{2z'}\right\},$$

then

$$\frac{\partial V}{\partial a}=\frac{\mu(a-x)}{z}+2k_1a+k_3a_1=0, \quad \frac{\partial V}{\partial a_1}=\frac{\mu'(a_1-x')}{z'}+k_3a+2k_2a_1,$$

$$\frac{\partial V}{\partial b}=\frac{\mu(b-y)}{z}+2k_1b+2k_3b_1=0, \quad \frac{\partial V}{\partial b_1}=\frac{\mu'(b_1-y')}{z'}+k_3b+2k_2b_1,$$

i.e.

$$-\frac{\mu(a-x)}{z}=+(2k_1a+k_3a_1) \quad \text{.....(1)}$$

$$-\frac{\mu'(a_1-x')}{z'}=+(2k_2a_1+k_3a), \quad \text{.....(2)}$$

or

$$a\left(2k_1+\frac{\mu}{z}\right)+k_3a_1=\frac{\mu x}{z}, \quad ak_3+a_1\left(2k_2+\frac{\mu'}{z'}\right)=\frac{\mu'x'}{z'},$$

$$a_1\left\{k_3^2-\left(2k_1+\frac{\mu}{z}\right)\left(2k_2+\frac{\mu'}{z'}\right)\right\}=\frac{\mu k_3x}{z}-\left(2k_1+\frac{\mu}{z}\right)\frac{x'\mu'}{z'}, \quad \text{.....(3)}$$

$$a\left\{k_3^2-\left(2k_1+\frac{\mu}{z}\right)\left(2k_2+\frac{\mu'}{z'}\right)\right\}=\frac{k_3\mu'x'}{z'}-\left(2k_2+\frac{\mu'}{z'}\right)\frac{\mu x}{z}. \quad \text{.....(4)}$$

$$\left[\text{Write } D \text{ for } -\left(2k_1+\frac{\mu}{z}\right)\left(2k_2+\frac{\mu'}{z'}\right)+k_3^2\right]$$

Similarly for b and b_1 in terms of y and y_1.

Substituting an amended form of V we obtain V as a function of xy, $x'y'$, zz' as follows:

$$V=k_0+\mu z+\mu'z'+k_1'(x^2+y^2)+k_2'(x'^2+y'^2)+k_3'(xx'+yy'),$$

where the new constants are readily expressed in terms of the old; for

$$(a_1-x')D=\frac{\mu k_3x}{z}-\frac{2k_1\mu'x'}{z'}-\frac{\mu\mu'x'}{zz'}-k_3^2x'+x'\left(2k_1+\frac{\mu}{z}\right)\left(2k_2+\frac{\mu'}{z'}\right)$$

$$=\frac{\mu k_3x}{z}+x'\left\{2k_2\left(2k_1+\frac{\mu}{z}\right)-k_3^2\right\},$$

$$(a-x)D=\frac{k_3\mu'x'}{z'}+x\left\{2k_1\left(2k_2+\frac{\mu'}{z'}\right)-k_3^2\right\}.$$

Now V may be written:

$$V=k_0+\mu z+\mu'z'+\tfrac{1}{2}a\left\{\frac{\mu}{z}(a-x)+2k_1a+k_3a_1\right\}+\tfrac{1}{2}b, \text{ etc.},$$

$$+\tfrac{1}{2}a_1\left\{\frac{\mu'}{z'}(a_1-x')+2k_2a_1+k_3a\right\}+\tfrac{1}{2}b_1, \text{ etc.},$$

$$+\tfrac{1}{2}x(x-a)\frac{\mu}{z}+\tfrac{1}{2}x'(x'-a)\frac{\mu'}{z'},$$

i.e.

$$V=k_0+\mu z+\mu'z'+\frac{\mu}{2z}x(x-a)+\frac{\mu'}{2z'}x'(x'-a)+\text{terms in } y,$$

$$=k_0+\mu z+\mu'z'+\frac{1}{2}\frac{\mu x}{D_1z}\left[\frac{k_3\mu'x'}{z'}+x\left\{2k_1\left(2k_2+\frac{\mu'}{z'}\right)-k_3^2\right\}\right]$$

$$+\frac{1}{2}\frac{\mu'x'}{z'D_1}\left[\frac{k_3\mu x}{z}+x'\left\{2k_2\left(2k_1+\frac{\mu}{z}\right)-k_3^2\right\}\right]$$

$$+\text{terms in } y$$

$$=k_0+\mu z+\mu'z'+\frac{1}{2}\frac{\mu}{D_1z}\left\{2k_1\left(2k_2+\frac{\mu'}{z'}\right)-k_3^2\right\}(x'+y^2)$$

$$+\frac{\mu\mu'k_3}{D_1zz'}(xx_1+yy_1)+\frac{1}{2}\frac{\mu'}{D_1z'}\left\{2k_2\left(2k_1+\frac{\mu}{z}\right)-k_3^2\right\}(x'^2+y'^2),$$

thus
$$k_1' = \frac{1}{2}\frac{\mu}{D_1 z^2}\left\{2k_1\left(2k_2+\frac{\mu'}{z'}\right)-k_3^2\right\},$$
$$k_2' = \frac{1}{2}\frac{\mu'}{D_1 z'}\left\{2k_2\left(2k_1+\frac{\mu}{z}\right)-k_3^2\right\},$$
$$k_3' = \frac{\mu\mu' k_3}{D_1 zz'}. \quad [D_1 = -D]$$

Now it can be shown $\dfrac{k_3^2 - 4k_1k_2}{k_3}$ is an invariant $= K$.

Hence choose z' and z so that $k_1' = 0$, $k_2' = 0$,

i.e.
$$z = \frac{2k_1\mu'}{k_3^2 - 4k_1k_2}, \quad z' = \frac{2k_2\mu}{k_3^2 - 4k_1k_2}.$$

With this shifting of the two origins by
$$\frac{2k_1\mu'}{k_3K}, \quad \frac{2k_2\mu}{k_3K}.$$
the terms in x^2+y^2 and $x'^2+y'^2$ disappear, and we have the simple form $k_0 + k''(xx'+yy')$ as the reduced path from $(x, y, 0)$ in one focal plane to $(x'y'.0)$ in the other.

The reduced path from (x, y, z) to $(x'y'z')$, the principal foci being the origins, will be obtained by starting with $k_1 = 0$, $k_2 = 0$, then the new coefficients are as before:
$$k_1' = -\frac{1}{2}\frac{\mu k_3^2}{z\left(k_3^2 - \dfrac{\mu\mu'}{zz'}\right)} = \frac{1}{2}\frac{\mu z'}{\left(-\dfrac{\mu\mu'}{k_3^2}+zz'\right)},$$
$$k_2' = \frac{1}{2}\frac{\mu' z}{\left(zz' - \dfrac{\mu\mu'}{k_3^2}\right)},$$
$$k_3' = \frac{\mu\mu' k_3}{zz'}\frac{1}{\left(\dfrac{\mu\mu'}{zz'} - k_3^2\right)} = \frac{\mu\mu'}{k_3 zz'} \times \frac{zz'}{\left(\dfrac{\mu\mu'}{k_3^2} - zz'\right)} = \frac{\mu\mu'}{k_3\left(\dfrac{\mu\mu'}{k_3^2} - zz'\right)}.$$

Hence function
$$V = k_0 + \mu z + \mu' z' + \frac{1}{2}\frac{\mu z'(x^2+y^2) - 2\mu f'(xx'+yy') + \mu' z(x'^2+y'^2)}{zz' - ff'}$$

where
$$f = \frac{\mu}{k_3}, \quad f' = \frac{\mu'}{k_3}.$$

Also equations 3, 4, 1, 2 of III. are now
$$a_1\left\{k_3^2 - \frac{\mu\mu'}{zz'}\right\} = \frac{\mu k_3 x}{z} - \frac{\mu\mu' x'}{zz'},$$
$$a\left\{k_3^2 - \frac{\mu\mu'}{zz'}\right\} = \frac{\mu' k_3 x'}{z'} - \frac{\mu\mu' x}{zz'},$$
$$k_3 a_1 = -\frac{\mu(a-x)}{z} = -\mu l,$$
$$k_3 a = \mu' l'.$$

If
$$z = \frac{\mu}{k_3},$$
$$z' = \frac{\mu'}{k_3},$$
$$x = a + a_1.$$

If $l=m=0$, $a_1=0$, $b_1=0$, *i.e.* the emergent ray passes through second principal focus, whence

$$a-x=lz, \qquad x'-a_1=l'z',$$
$$x=a-lz, \qquad x'=a_1+l'z',$$
$$=\frac{\mu' l'}{k_3}-lz, \qquad \text{i.e.} \quad x'=l'z'-\frac{\mu l}{k_3}.$$

If $$z=\frac{\mu}{k_3} \quad \text{and} \quad z'=\frac{\mu'}{k_3},$$
$$x=\frac{\mu' l'}{k_3}-\frac{\mu l}{k_3}, \qquad x'=\frac{\mu' l'}{k_3}-\frac{\mu l}{k_3},$$

i.e. $$x=x'.$$

Similarly $$y=y'.$$

Hence $z=f$ and $z=f'$ are the unit planes, and f and f' the focal lengths.

If $$z=f'=\frac{\mu'}{k_3} \quad \text{and} \quad z'=f=\frac{\mu}{k_3},$$

$x=0=x'$ for any equal values of l and l'. Similarly $y=0=y'$.

Hence any incident ray crossing the axis at the first nodal point crosses the axis on emergence at the second focal point [nodal points are the unit points if the first and last media are the same].

We have thus seen how the reduced path between two points may be expressed in terms of the coordinates of those points and involving constants depending on the curvatures, thicknesses and separations of the surfaces composing the optical system. We now have to face the question of determining these constants for a given system.

IV. *The determination of the constants.*

Let the surfaces have as approximate equations $2\rho\zeta=\xi^2+\eta^2$, *i.e.* take the intersection of the axis and each surface as the origin for that surface.

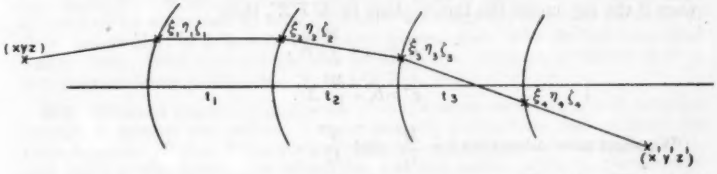

FIG. 13.

The path from (x, y, z) to $(x'y'z')$,

$$\mu_1\{(\xi_1-x)^2+(\eta_1-y)^2+(\zeta_1-z)^2\}^{\frac{1}{2}}+\mu_2\{(\xi_2-\xi_1)^2+(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2\}^{\frac{1}{2}}$$
$$+\ldots+\mu_n\{(\xi_n-x')^2+(\eta_n-y')^2+(\zeta_n-z')^2\}^{\frac{1}{2}}.$$

This may be written correct to the second order of small quantities.

$$U=\mu_1\left\{z+\frac{x^2+y^2}{2z}-\frac{\xi_1 x+\eta_1 y}{z}+\frac{\xi_1^2+\eta_1^2}{2}\left(\frac{1}{z}-\frac{1}{\rho_1}\right)\right\}$$
$$+\mu_2\left\{t_1+\frac{\xi_1^2+y_1^2}{2}\left(\frac{1}{\rho_1}+\frac{1}{t_1}\right)+\frac{\xi_2^2+y_2^2}{2}\left(\frac{1}{t_1}-\frac{1}{\rho_2}\right)\right\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\mu_n\left\{z'+\frac{\xi_n^2+\eta_n^2}{z}\left(\frac{1}{\rho_2}+\frac{1}{z'}\right)+\frac{x'^2+y'^2}{2z'}-\frac{\xi_n x'+\eta_n y'}{z'}\right\}.$$

By Fermat's theorem $\frac{\partial U}{\partial \xi}=0=\frac{\partial U}{\partial \eta}$ for all suffixes giving sufficient equations to solve for the ξ's and η's. Substituting we have the reduced path again in terms of the coordinates but with the correct constants.

V. *The aberrations.*

Clerk Maxwell then went on to discuss the aberrations by including the next order of small quantities so that V becomes

$$V=\mu\left\{z+\frac{(a-x)^2+(b-y)^2}{2z}\right\}+k_0+k_1(a^2+b^2)+k_2(a_1^2+b_1^2)+k_3(aa_1+bb_1)$$
$$+k_4(a^2+b^2)^2+k_5(a_1^2+b_1^2)^2+k_6(aa_1+bb_1)^2+k_7(a^2+b^2)(a_1^2+b_1^2)$$
$$+k_8(aa_1+bb_1)(a^2+b^2)+k_9(aa_1+bb_1)(a_1^2+b_1^2)$$
$$+\mu'\left\{z'+\frac{(a_1-x')^2+(b_1-y')^2}{2z'}\right\}.*$$

If we agree that the reduced path shall be from a point on the xy plane (a, b) to a point (a_1b_1) in the image xy plane, then

$$V=k_0+k_1(a^2+b^2)+\ldots+k_9(aa_1+bb_1)(a_1^2+b_1^2),$$

and if the coordinate planes are through the principal foci, then k_1 and k_2 are also zero. Taking the former of these two, and taking L', M', N' as the direction cosine of the emergent ray,

$$M'=\frac{\partial V}{\partial a_1}=2k_2a_1+k_3a_1+2k_5a_1(a_1^2+b_1^2)+2k_6a(aa_1+bb_1)$$
$$+2k_7a_1(a^2+b^2)+k_8a(a^2+b^2)+2k_9a_1(aa_1+bb_1)+k_9a(a_1^2+b_1^2),$$

$$N'=\frac{\partial V}{\partial b_1}=\text{similar expression},$$

$$L'=\sqrt{1-M'^2-N'^2};$$

then if the ray meets the image plane in $X'Y'Z'$, then

$$Y'=a_1+\frac{M'}{L'}X',$$

$$Z'=b_1+\frac{N'}{L'}X'.$$

We must now substitute for $\frac{M'}{L'}$ and $\frac{N'}{L'}$,

$$\frac{M'}{L'}=\frac{M'}{\sqrt{1-M'^2-N'^2}}=M'\{1+\tfrac{1}{2}(M'^2+N'^2)\}=M'+\tfrac{1}{2}M'^3+\tfrac{1}{2}M'N'^2.$$

Similarly $$\frac{N'}{L'}=N'+\tfrac{1}{2}M'^2N'+\tfrac{1}{2}N'^3.$$

Substituting for M' and N' their values and retaining only third order terms, we have finally

$$Y'=a_1+X'\{2k_2a_1+k_3a+\text{3rd order terms}\},$$
$$Z'=b_1+X'\{2k_2b_1+\text{3rd order terms}\}$$

simplifying the work by making $b=0$,

$X'=-\frac{1}{2k_2}$ makes Y' proportional to a and $Z'=0$, neglecting 3rd order terms, so that we have finally

* *Phil. Mag.*, March 1929. A. Buxton.

δY the displacement due to the inclusion of the 3rd order terms

$$= -\frac{1}{2k_2}\left\{\begin{array}{l}(a_1(a_1^2+b_1^2)(2k_5+4k_2^3)^*+2a_1a^2(k_6+k_7+\tfrac{3}{2}k_2k_3^2)\dagger\\ +aa_1^2(3k_9+6k_3k_2^2)+ab_1^2(k_9+2k_3k_2^2)+a^3(k_2^3+k_8)\ddagger\end{array}\right\},$$

$$\delta Z = -\frac{1}{2k_2}\left\{\begin{array}{l}b_1(a_1^2+b_1^2)(2k_5+4k_2^3)^*+(2k_7+k_2k_3^2)a^2b_1\dagger\\ +(2k_9+4k_2^2k_3)(aa_1b_1)\end{array}\right\}.$$

Finally one has to face the problem of determining the constants for any particular system even when the 3rd order terms are included. This means evaluating the k's up to k_9. Various methods have been suggested for the single surface and the thin lens (see Herman's *Geometrical Optics*, ch. xiv.), and Clerk Maxwell (see vol. ii of his *Collected Papers*) has dealt with the Curvature of the Image. Much, however, remains to be done for the general problem when a large number of surfaces compose the system and when the fifth order terms are taken into account.§

A. BUXTON.

GLEANINGS FAR AND NEAR.

698. *Swift and Leap-year.* The following occurs in the *Journal to Stella*, March 1, 1710-11 :

"*Morning.* I have been calling to Patrick to look in his almanac for the day of the month ; I did not know but it might be leap-year. The almanac says it is the third after leap-year, and I always thought till now that every third year was leap-year. I am glad they come so seldom ; but I am sure it was otherwise when I was a young man : I see times are mightily changed since then."

Swift did not pick up much ordinary school learning while he was young ; but the above is almost beyond comprehension. That he had a good head for figures, and for expressing proportions in numbers, any one who has been with him to Lilliput and Brobdignag will not fail to see. Possibly he might have picked up his notion in this way. Say that in 1679-80 he happened to see the almanac (which counted 1680 from January 1, as did all the almanacs) from which he would learn that 1680 is leap-year. Suppose that in 1683-84 he happened to note February 29, from the common parlance of those about him, as falling in 1683, and to remember that the last leap-year was in 1680. With such a departure he might live in the belief that leap-year comes every three years.—M. (*N. and Q.* I. x. 242).

699. Sylvester enjoyed stimulants. I do not mean such vulgar and material articles as alcohol and coffee. I never saw any indications that he cared for their support. But he loved such stimulants to intellectual activity as music, and light lively society, in which he was not called upon to participate. Once at a symphony concert I sat just behind him, admiring the dome of his capacious cranium, unconcealed by hair, and noticed how absorbed he was. The next day, Sunday, he came to me impetuously to say that he had worked out some mathematical proposition at the concert of the evening before, the concert having quickened his mathematical mind. He really thought that this was his greatest achievement yet, and he had hastened to write it out and mail it to the Academy of Sciences in Paris. Once he told me that, having a special paper to prepare, he went to a store and bought a pound of candles, which he placed about his room, on all sorts of extemporaneous candlesticks ; "for light," he said, "is a most powerful tonic."—Dr. Gilman, of Johns Hopkins University, quoted in *The Popular Science Monthly*, c. 1903.

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* Spherical aberration. † Curvature and astigmatism. ‡ Distortion. With coma in the remaining terms.

§ Schwartzschild.

THE SIEVE OF ERATOSTHENES.

By R. A. Fisher, F.R.S.

I am too ignorant of the literature to know what views have been actually expressed by moderns as to the famous sieve of Eratosthenes, yet I think Sir Thomas Heath may be relied on to mention any which seem to throw light upon early Greek mathematics. His description (*Greek Mathematics*, I. 100) from Nicomachus, 100 A.D., whom there is no reason to regard as a mathematician, is as follows:

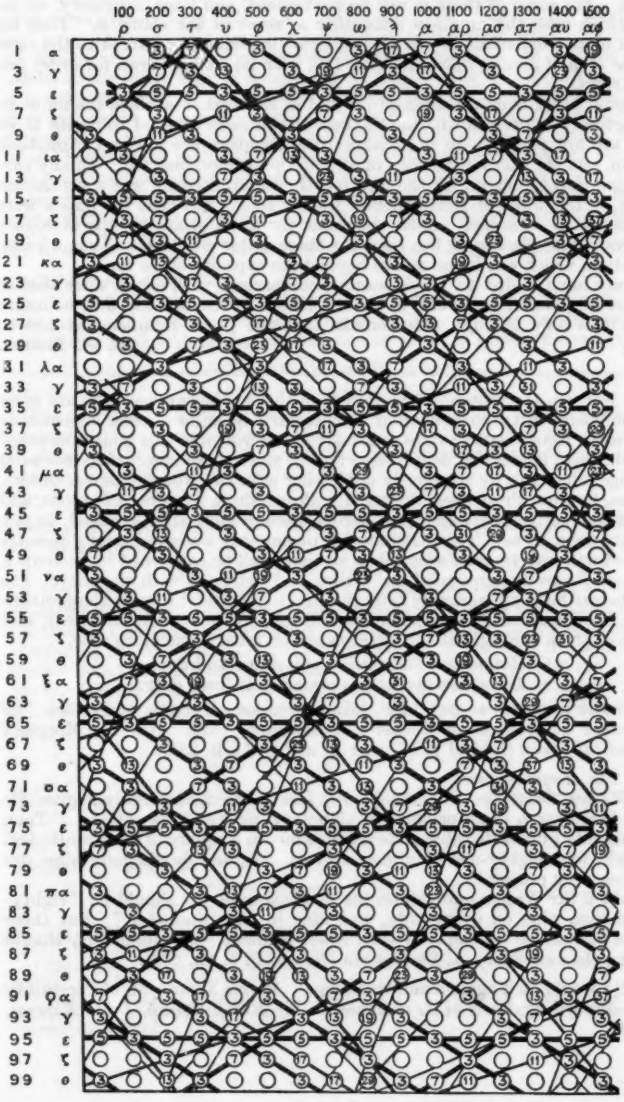
> The method is this. We set out the series of odd numbers beginning from 3.
>
> 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, ...
>
> Now 3 is a prime number, but multiples of 3 are not; these multiples, 9, 15, ... are got by passing over two numbers at a time beginning from 3; we therefore strike out these numbers as not being prime. Similarly 5 is a prime number, but by passing over four numbers at a time, beginning from 5, we get multiples of 5, namely 15, 25 ...; we accordingly strike out all these multiples of 5. In general if n be a prime number, its multiples appearing in the series are found by passing over $n-1$ terms at a time, beginning from n; and we can strike out all these multiples. When we have gone far enough with this process, the numbers which are still left will be primes. Clearly, however, in order to make sure that the odd number $2n+1$ in the series is prime we should have to try all the prime divisors between 3 and $\sqrt{(2n+1)}$; it is obvious, therefore, that this primitive empirical method would be hopeless as a practical means of obtaining prime numbers of any considerable size.

One must feel some surprise that a mathematical contemporary of Archimedes, of great repute in his time—he was nicknamed *pentathlos* from the diversity of his attainments—in Athens and in Alexandria, should have his name associated, with so feeble and useless a device, as though with some considerable or specially ingenious discovery. Heath's words "primitive" and "empirical" are exceedingly moderate in criticism of the method of constructing tables by successive addition, of the multiples of all primes less than $\sqrt{n}$, in order to find out if n is prime.

It is perhaps worth while to ignore the description, which reads like a portion of a demonstration, and attend to the name. The sieve is a machine, and therefore a contrivance to save labour. It is not designed to determine the properties of any individual object, for which we might use a measure or a gauge, but to sort out automatically from a bulk of mixed material, that portion which has some particular property. The job is done once for all; and if we may assume that the sieve of Eratosthenes was named by men who understood its purpose, and that the device was thought sufficiently ingenious, or the name sufficiently witty, to be memorable, we may perhaps take it for granted that its purpose was not to determine whether a particular number was a prime, but to sort out the whole of the primes, more rapidly than could otherwise be done, from an extensive series of consecutive numbers. The device would then work like a sieve. If it also looked like a sieve we should the better understand how it was that the name struck the Greek fancy.

The diagram (page 565) shows the 800 odd numbers less than 1600, arranged in 16 columns of 50 each. Since 102 is an even number divisible by 3, we may pass from any one odd multiple of 3 to another by moving to the next column and taking one step of 2 units down, and the continuation of the oblique line joining these two numbers will pass through nothing but numbers divisible by 3, as far as the edge of the diagram. The series of such oblique lines, which automatically strike out the numbers divisible by 3, are equally spaced at intervals of 3 odd numbers from the top of the diagram to the bottom, or

RECONSTRUCTION OF THE SIEVE OF ERATOSTHENES



perhaps one should say from its N.E. to its S.W. corner, and may be drawn without hesitation without the tedium of counting 3, 530 times or so.

When the threes are complete we can put in the lines for 5, which happen to be horizontal, for 7, which pass obliquely to the N.E., and already build up a tolerable picture of a wicker sieve, and continue the process for the higher primes. For some of these it will be convenient, and later obligatory, to use lines which miss the numbers altogether in some of the columns. Thus for 11 it is more convenient to use the multiple 198, entailing 2 steps to the right and one step of 2 units upwards, than 110, with its 5 steps down for every one to the right.

I should judge from the description that in Eratosthenes' sieve the numbers were actually written down, and the bars of the sieve allowed to delete those which are not primes. In my diagram the numbers are inferred from their position, and the smallest factors, of all but the primes, inserted. In any case I imagine that the sieve, for the first few thousand integers, was a permanent feature on the walls of Eratosthenes' lecture room, and that his pupils were encouraged to construct similar diagrams for appropriate groups of higher numbers. The placing of the lines for these latter would entail some initial calculation, which was by no means beyond the capacity of his time.

It may have added to the humorous comparison with a real sieve that of each set of parallel bars, one is broken before it reaches the first column; indeed it is only through these breaks that the prime numbers manage to escape.

R. A. FISHER.

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700. For you the pretended Numerists of this vapouring age, who are more disingeniously witty to propound unnecessary questions, than ingeniously judicious to resolve such as are necessary. For you was this book composed and published, if you will deny yourselves so much as to invert the streams of your ingenuity, and by studiously conferring with the Notes, Names, Orders, Progress, Species, Properties, Proprieties, Proportions, Powers, Affections, and Applications of Numbers delivered herein, become such artists indeed, as you now only seem to be. This Arithmetick ingeniously observed, and diligently practised will turn to good account to all that shall be concerned in Accompts. All whose Rules are grounded on Verity, and deliver'd with sincerity. The examples are built up gradually from the smallest consideration to the greatest. All the Problemes or Propositions are well weigh'd, pertinent and clear, and not one of them throughout the Tract taken on trust; therefore now,

'Zoilus and Momus lye you down and dye,
For these inventions your whole force defy.'

From (what professes to be) "Mr. Edward Cocker's Proeme or Preface," to the Second Impression of Cocker's *Arithmetick*, 1679. (Cf. De Morgan's *Arithmetical Books* (foot of p. 57) and *N. and Q.* II. ii. 252.)

701. *The Magic Square.*

Sir Walter Scott, in *The Antiquary*, has made Dousterswivel give Sir Arthur Wardour a description of a planetary *sigil* as it is called: "Then upon this side I make de table of de moon, which is a square of nine, multiplied into itself, with eighty-one numbers on every side, and de diameter nine, der it is, done very proper."

I doubt if Sir Walter understood the magic square. It should be "eighty-one numbers, nine on every side, and nine in each diameter." But these numbers should be so disposed that rows, columns, and diameters, should each give the sum 369.—A. De Morgan (*N. and Q.* I. xii. 238).

702. "I would do it again a hundred times. Better X per cent. of a certainty than X divided by 1 per cent. of chaos."—Sir Josiah Stamp, *Daily Express*, 5, ix. 29.

MAGNITUDE AND MINUTENESS—A CRUX IN CALCULATION INVOLVING THE USE OF BINOMIAL SERIES—ARISING OUT OF A PROBLEM IN PROBABILITY.*

By Prof. P. J. Heawood, M.A.

This is a clumsily long title, but it might have been much more lengthy if it had gone on to suggest how the problem in probability arose. That would involve us in many questions, so we may say at once that the problem is equivalent to the following: If from a bag containing M different balls N drawings are made, where M, N are very large numbers (the ball being replaced after each drawing), what is approximately the chance that exactly r balls should fail to appear? In the crucial case with which I was concerned, which I hasten to add had nothing really to do with bags or balls—that is merely a convenient and easily understood equivalent—$M=3^{13}=1,\ 594323$; $N=4^{12}=16,\ 777216$.† Putting $N=\mu M$, μ is a little more than $10\frac{1}{2}$, so that on the average each ball might appear about 10 times. Now the chance that r *specified* balls should fail to appear is $\left(\frac{M-r}{M}\right)^N$, since at each draw there are $M-r$ out of M which are available: call this P_r.

2. Now if, of a number of events that may happen, $\Sigma p_{(r)}$ is the *sum* of the chances for all possible *specified* sets of r to happen, the chance that *some* r events *exactly* should happen, no less and no more, is given by the formula

$$c_0\Sigma p_{(r)}+c_1\Sigma p_{(r+1)}+c_2\Sigma p_{(r+2)}+\dots,$$

where

$$(1+x)^{-r-1}\equiv c_0+c_1x+c_2x^2+\dots.$$

The formula may be verified in this way. Consider a case where *some* s exactly of the events happen. These s will include ${}_sC_r$ different sets of r and this case will be counted ${}_sC_r$ times in the reckoning for $\Sigma p_{(r)}$, ${}_sC_{r+1}$ times in the reckoning for $\Sigma p_{(r+1)}$ and so on, and therefore

$$c_0\cdot{}_sC_r+c_1\cdot{}_sC_{r+1}+c_2\cdot{}_sC_{r+2}+\dots$$

times in the reckoning of the above formula. But this is the coefficient of x^{s-r} in the product

$$(c_0+c_1x+c_2x^2+\dots)({}_sC_0x^s+{}_sC_1x^{s-1}+\dots),$$

i.e. in $(1+x)^{s-r-1}$; *i.e.* it is *unity* if $s=r$, otherwise zero. Thus a case where exactly r of the events happen comes into the reckoning once and once only: no other is counted (on the whole). Therefore the formula gives correctly the chance for *exactly* r events. But our business is rather with the application of the formula than with any elaborate proof of it. In our use of it the "events" are the failures of the different balls to appear in the course of N drawings and, since there are ${}_MC_m$ different sets of m balls, and the chance of failure for each set is P_m,

$$\Sigma p_{(m)}={}_MC_m\cdot P_m={}_MC_m\left(1-\frac{m}{M}\right)^N.$$

Substituting in the formula, we have for the chance of exactly r failures

$$\sum_{s=0}^{M-r}c_s\cdot{}_MC_{r+s}\left(1-\frac{r+s}{M}\right)^N,$$

which is the same as

$$\sum_{s=0}^{M-r}(-1)^s\,{}_MC_r\cdot{}_{M-r}C_s\left(1-\frac{r+s}{M}\right)^{\mu M},$$

(writing μM for N), as is easily verified.

* The substance of a paper read to the North-Eastern Branch of the Mathematical Association on May 4th, 1929.

† This case is one of a set for which $M=3^m$, $N=4^{m-1}$; but one to which special interest attached, both in itself and as perhaps the first to which the following approximation could be successfully applied.

3. For moderate values of M, N this might suffice; but for large values of M the summation is not so easily effected, even to a limited number of terms. For moderate values of r, s however, where M is large [and we shall see abundantly in the sequel that when $r+s$ is considerable the terms are so small that they may be left out of account] there is a tempting approximation possible. We have:

$$\left(1-\frac{r+s}{M}\right)^{\mu M}=\left\{\left(1-\frac{r+s}{M}\right)^{-\frac{M}{r+s}}\right\}^{-\mu(r+s)},$$

and supposing $M/(r+s)$ to be large, this is approximately equal to $e^{-\mu(r+s)}$. Accepting this value and putting separately factors which do not involve s, our formula becomes:

$$e^{-\mu r}\ {}_MC_r \sum_{s=0}^{M-r}(-1)^s\ {}_{M-r}C_s\ e^{-\mu s},$$

which has the value $\qquad e^{-\mu r}\ {}_MC_r\,.\,(1-e^{-\mu})^{M-r}$;

the terms of the Σ, alternately positive and negative, being those of the binomial expansion of $(1-e^{-\mu})^{M-r}$, which we shall refer to as a subsidiary series. The result may be written in the slightly different form:

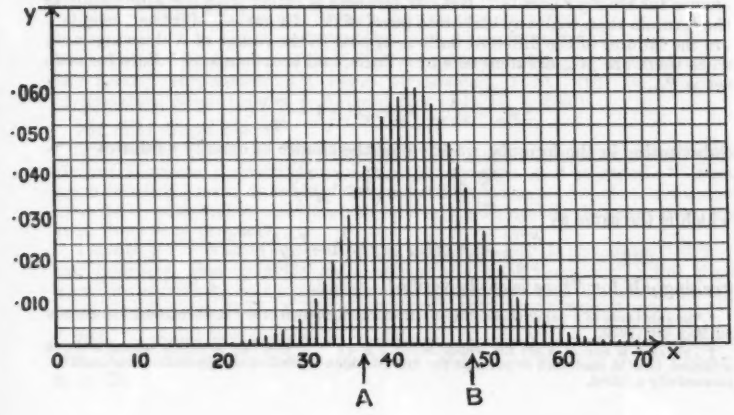
$$(1-e^{-\mu})^M\,.\,{}_MC_r(e^{\mu}-1)^{-r}.$$

In this form the chances, for different values of r, are seen to be proportional to the terms of a unique binomial series, this time with all its terms positive, that for $\{1+(e^{\mu}-1)^{-1}\}^M$.

It is very remarkable that, though the formula can be only approximate, the sum of the chances given by it for all values of r from 0 to M works out to exactly unity, as it should; for

$$(1-e^{-\mu})^M\{1+(e^{\mu}-1)^{-1}\}^M=1.$$

4. Coming now to actual arithmetic, our result, with the values of M and N which were proposed, makes the different chances equal to the terms of a binomial expansion $\equiv(1+x)^M$, multiplied by a certain multiplier; where $x=\cdot00002690844$, while $M=1,594323$; and it is of some interest in itself to observe how a binomial series behaves for such large and small values. The multiplier $(1-e^{-\mu})^M=2\cdot3\ldots\times10^{-19}$, and it is in virtue of its extreme minuteness that the larger terms are brought within manageable compass. The graph below indicates the values of the terms, as thus multiplied, for a small range at the beginning of the series, and shows very clearly how rapidly they increase, up to $r=42$, and how soon they die away into comparative nothingness; this



is the more striking when we remember that the complete series includes more than a million and a half terms and that on this small scale the complete graph would extend to well over a mile! The sum of 100 terms to 8 places works out to ·99999999, while (as we have seen) the sum of the whole is exactly unity, so minute is the total value of the last million and a half terms.

5. But we must recur to the consideration of the subsidiary binomial series, with terms alternately positive and negative, or rather the succession of such series for different values of r, which formed the basis of our calculations. The value of such a series $(1-e^{-\mu})^{M-r}$, multiplied by $e^{-\mu r}{}_{M}C_{r}$ was, perhaps somewhat hastily, adopted as giving a sufficiently good approximation to the chance of r omissions. Curiously the *numerical values* of its terms, when r is inconsiderable, differ very slightly from those of the later series $\equiv\{1+(e^{\mu}-1)^{-1}\}^{M}$; for $M-r$ is comparable with M and $e^{-\mu}$ hardly distinguishable from $(e^{\mu}-1)^{-1}$. In fact the same graph, which was introduced to exhibit the final result, may be taken as showing (on the appropriate scale) the *numerical values* of the terms of such a subsidiary series. These values, it will be remembered, were arrived at by assuming that for moderate values of r and s we might substitute $e^{-\mu(r+s)}$ for $\left(1-\frac{r+s}{M}\right)^{\mu M}$.

To what extent is this justified?

We have $$\log\left(1-\frac{r+s}{M}\right)^{M}=-\mu M\left\{\frac{r+s}{M}+\frac{(r+s)^2}{2M^2}+\dots\right\}$$
$$=-\mu(r+s)-\mu\frac{(r+s)^2}{2M}-\dots$$

Therefore $$\left(1-\frac{r+s}{M}\right)^{\mu M}=e^{-\mu(r+s)}\,e^{-\mu\frac{(r+s)^2}{2M}\dots}$$
$$=e^{-\mu(r+s)}\left\{1-\mu\frac{(r+s)^2}{2M}-\dots\right\}$$

i.e., neglecting the less important terms, the value adopted was in excess by some $\mu\frac{(r+s)^2}{2M}$ of itself.

Take *e.g.* the case of $r=36$, where as our graph shows the chance is high, and take the most important term of the subsidiary series for that value of r, *i.e.* $s=42$ (as illustrated by the graph in its subsidiary application), then $\mu\frac{(r+s)^2}{2M}=\frac{78^2\mu}{2M}=\frac{1}{50}$ nearly, a proportionate error which might not seem very serious.

But the whole situation is changed when we remember that the terms are here alternately positive and negative and that the magnitude of any errors must be weighed not against the individual terms in which they occur but against the whole value of the series. When we take this into account the result is at first sight appalling. Calculation shows that the term we have been considering—the 43rd of $(1-e^{-\mu})^{M-36}=10^{17}\times 2\cdot 6\dots$, a number of 18 figures; and the relatively small error of $\frac{1}{50}$ of that term $=10^{15}\times 5\cdot 2\dots$ a number of 16 figures. On the other hand the whole value of the binomial series $(1-e^{-\mu})^{M-36}$ is $\cdot 0^{15}23394$, with eighteen zeros after the decimal point, so that the error in the individual term, which, relatively to the value of that term, might be lightly regarded, is something like 10^{34} times the value of the whole series. So easy is it to burn one's fingers in dealing with such magnitudes.

6. This result might seem to make the whole approximation hopeless; but it must be remembered that if the positive and negative terms of the binomial series are so nicely balanced, it is possible that the errors may be so likewise. To show that they are even more nicely balanced and practically cancel out is the burden of this paper.

Writing M' for $M-r$, the error we are considering in an individual term of the alternating binomial expansion amounts to

$$\frac{\mu}{2M}(r+s)^2(-1)^s{}_{M'}C_s e^{-\mu s}$$

$$=\frac{\mu}{2M}\left\{r^2(-1)^s{}_{M'}C_s e^{-\mu s}+(2r+1)s(-1)^s{}_{M'}C_s e^{-\mu s}+s(s-1)(-1)^s{}_{M'}C_s e^{-\mu s}\right\}$$

$$=\frac{\mu}{2M}\left\{r^2(-1)^s{}_{M'}C_s e^{-\mu s}-(2r+1)M'e^{-\mu}.(-1)^{s-1}{}_{M'-1}C_{s-1}e^{-\mu(s-1)}\right.$$

$$\left.+M'(M'-1)e^{-2\mu}.(-1)^{s-2}{}_{M'-2}C_{s-2}e^{-\mu(s-2)}\right\}.$$

Summing for s from 0 to M' we have for the combined effect of all the errors*

$$\frac{\mu}{2M}\left\{r^2(1-e^{-\mu})^{M'}-(2r+1)M'e^{-\mu}(1-e^{-\mu})^{M'-1}\right.$$

$$\left.+M'(M'-1)e^{-2\mu}(1-e^{-\mu})^{M'-2}\right\}.$$

Putting k for $e^{-\mu}/1-e^{-\mu}$ and dividing out by $(1-e^{-\mu})^{M'}$, remembering also that $M'=M-r$, the total correction required

$$=\frac{\mu}{2M}\left\{r^2-(2r+1)(M-r)k+(M-r)(M-r-1)k^2\right\}$$

of the whole value. The fraction is thus a quadratic in r

$$=\frac{-\mu(k+1)^2}{2M}(\alpha-r)(\beta-r)\,(\text{say})\,\dagger$$

where, in the case with which we are dealing, the multiplier

$$=-\cdot00000330035\,;\ \alpha=49\cdot4493,\ \beta=36\cdot3499.$$

The correction is therefore positive for values of r between 37 and 49 inclusive, otherwise negative; *i.e.*, recurring to the graph in its original application, the correction is positive for values between the limits marked A and B and negative beyond those limits. And for the important terms between these limits the maximum *fractional* error is only about ·00014, or say $\frac{1}{7000}$, of the value in question. That the correction should be sometimes + and sometimes − is inevitable, because (as remarked before) the sum of all the chances, as given by our approximate formula, is (rightly) unity.

Thus, in spite of errors in the individual terms of a subsidiary series which are stupendous compared with the value of the whole, the total effect is a quite inappreciable fraction of that value and our formula is vindicated after all. The whole investigation shows both the care which is needed in dealing with very large and very small quantities and the interesting developments to which nevertheless they lend themselves. We are often impressed by the magnitude and minuteness of the quantities which present themselves in Physics and Astronomy; but applications of the elementary formulae of pure mathematics furnish instances, no less striking, where we have to deal with the very great or the very small.

Durham, July 1929. PERCY J. HEAWOOD.

* A difficulty may be felt from the fact that the expression for the error is appropriate only when $r+s$ is inconsiderable. The answer is that the terms rapidly become so small that they may be left out of account altogether. We only need in effect the terms from $s=0$ to $s=s'$, where s' is quite moderate, say 160 in the case in point. So in summing for the errors we may stop at the 160th term; but there again the sum from 0 to s' will not be appreciably different from the sum from 0 to M' which gives the simple result above.

† A like method applies to the less important errors expressed by subsequent terms in the expansion of $(1-(r+s)/M)^{\mu M}$ (§ 5); only they involve higher powers of r. Thus for the next in importance we have a cubic in r, with more complicated maxima and minima, but the general effect will be very much less.

ON THE TEACHING OF ANALYSIS

By E. G. Phillips, M.A., M.Sc.

Every attempt to give an elementary rigorous course of Analysis is beset by several unsurmountable difficulties. Every teacher of the subject will readily agree that the two main obstacles are (i) Dedekind's theorem, and (ii) the Heine-Borel theorem. Of these the first may be rendered less apparent, for in an *elementary* treatment there is no great objection to the postulation of the existence of an upper and a lower bound for every bounded linear set of numbers. The second cannot, however, be avoided, for the important theorems of the differential calculus, such as Rolle's theorem, the Mean-value theorem, Cauchy's formula, and Taylor's theorem can only be rigorously proved by appealing to certain properties of continuous functions, which in their turn depend upon the Heine-Borel theorem, or upon some alternative simpler theorem on intervals such as the one which Carslaw * uses to replace it.

These are not, however, the only difficulties. The so-called "elementary" functions, $\sin x$, $\cos x$, $\log x$ and e^x are all introduced in elementary text-books by methods which involve an appeal to geometrical intuition as an element of proof. It is in fact impossible to give a rigorous proof that the derivative of $\sin x$ is $\cos x$ without appealing to the theory of term-by-term derivation of an infinite series, and this of course involves the concept of uniformity of convergence. The usual elementary proof,† which depends upon the result

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1,$$

which is deduced from the inequalities

$$\cos \theta < \frac{\sin \theta}{\theta} < 1,$$

must involve an appeal to geometry, and in any case it involves the notion of the length of a curve; and in spite of what many elementary books seem to suggest, this concept is not a simple one, even though the curve in question is "only a circle!"

Again, whichever of the two usual methods of defining the functions $\log x$ and e^x be adopted, either the concept of a Riemann integral or else some knowledge of operations on infinite series must be presupposed. Here again one is faced with the necessity for assuming another difficult "abstract" theorem, namely the existence theorem for inverse functions, which is so often tacitly assumed.

In my Honours lectures I have always found that the only satisfactory method of procedure is to start the subject afresh right from the beginning, and then develop it logically, starting with the concept of "number" and proceeding in what appears to me to be a natural sequence of steps.‡ I refer to this again later. Needless to say it is never very easy to convince the student whose previous training has been on quite different lines that many of his so-called "proofs" are no longer of any use at all!

If this kind of teaching could be started early enough in the students' college career all would be well, but the problem which I have had to face is a difficult one. According to the regulations governing the courses pursued by the mathematical Honours students in the University of Wales, all such students normally pursue the final year of the Pass degree course in Pure mathematics *concurrently* with the Honours course to which I have referred above. It is unavoidable that in the Analysis work a certain amount of overlapping should occur; and my problem has been to make as much of the Pass

* *Theory of Fourier Series and Integrals* (1921), § 18.

† See, for example, Gibson's *Calculus* (1919), pp. 77 and 129.

‡ The order of development of the subject which I prefer will be seen in a book which I have just completed, and which will shortly be published by the Cambridge University Press.

course as possible useful to the Honours students who attend it, without at the same time appearing too "rigorous" to alarm the others. It may interest some of my readers if I state briefly how part of this difficulty has been met. As "gently" as possible the concept of continuity is presented in the Pass course; then two theorems are assumed without proof, namely (1) *A continuous function is bounded*, and (2) *A continuous function attains its bounds*. It is, of course, easy to illustrate geometrically the truth of these theorems. By appealing to these two theorems, rigorous proofs of Rolle's theorem, the Mean-value theorem, Cauchy's formula and Taylor's theorem can be given, and then suitable applications of these theorems, including the evaluation of the limits of functions which assume the indeterminate form 0/0, readily follow. When the point is reached in the Honours course for introducing the above-named theorems, the student may be referred for the proofs to his Pass course. A similar method is adopted for dealing with differentiability of functions of more than one variable.

I now mention a few interesting points in connection with the Analysis portion of the Honours course in Pure Mathematics.* It may be a debatable question whether any mathematical philosophy should be included in a first Honours course, but if time allows, and if the students of any given year are good enough, it is interesting to start with a brief introduction to mathematical philosophy by first giving the logical definition of number, and then Russell's modification of Dedekind's method of defining an irrational number. This introduces a difficulty which may be worth mentioning. When Dedekind's definition is abandoned, and Russell's definition of a real number as a segment of the series of ratios† has been adopted, the usual method of proving, by an appeal to Dedekind's theorem, that any linear bounded set of numbers has definite bounds, needs replacing by some other device. What appears to be the best method of dealing with the aggregate of *real* numbers, so as to be able to introduce the concept of an (exact) upper and lower bound, is to define upper and lower segments of the aggregate of real numbers (in an analogous way to that of defining segments of the series of ratios), and then to assert what may conveniently be called the "Principle of Continuity" to the effect that *every segment of the aggregate of real numbers determines a definite unique real number*. The definition of the lower bound of a given set of numbers (x) is then as follows. Suppose that there exist numbers H and K, such that

$$H < x < K$$

for all values of x in the set: then H is a (rough) lower bound. The aggregate of rough lower bounds forms a lower segment, for if $H' < H$, then H' is another rough lower bound. The definite unique real number determined by the aggregate of rough lower bounds is the (exact) lower bound m of the set (x). Similarly the upper bound M may be defined.

A convenient and logical order of development of the subject is then briefly as follows. The concepts of the upper, lower and unique limits of a bounded sequence (s_n). The theory of limits and continuity of functions of a single real variable, with obvious extensions to multiple limits. The definitions of derivative, differential, and the fundamental theorems of the differential calculus. Some elementary theory of series of positive terms, followed by a discussion of Taylor's and Maclaurin's series. The Riemann integral and its properties. Differentiability of functions of more than one variable, first for explicit and then for implicit functions. At this stage the abstract existence theorems for implicit functions should be considered. The concept of uniformity of convergence and the theory of power series should be introduced

* In 1931 this will be superseded in Bangor by a joint Honours course in Pure and Applied mathematics, and so a good deal of the work now done in Pure mathematics may have to be abandoned.

† See *Introduction to Mathematical Philosophy*, p. 72.

into the course as early as possible. One great advantage of this is that it enables the students to make an intelligent use of the powerful weapons of term-by-term derivation and integration of an infinite series, while fully recognizing their limitations. The fact that these operations on *power series* are almost always valid is a great temptation for allowing them to be used before they can be justified.

The "elementary" functions are conveniently taken as known right from the commencement, although it is necessary to warn the student that their elementary properties are assumed, as suggested by geometrical intuition, for the chief reason that it is desirable to enrich our examples by the use of these functions as soon as possible. A rigorous definition of the logarithm by means of the equation

$$\log x = \int_1^x \frac{dt}{t} \quad (x > 0)$$

can be given after Riemann integration has been discussed, but the deduction of e^x as the inverse function of $\log x$ requires the existence theorem for implicit functions (for the special case of inverse functions). The trigonometrical functions $\sin x$ and $\cos x$, defined as the sum-functions of the two well-known infinite series, can only be rigorously discussed after the theory of power series has been considered. An alternative method of developing the trigonometrical functions, based on the equation

$$y = \arctan x = \int_0^x \frac{dt}{1+t^2}$$

is described by Hardy,* and this method has the advantage of being on the same lines as the definition of $\log x$, and so could be introduced at the same point in the course. Personally I prefer the definition by means of the infinite series.

The consideration of double, triple, curvilinear and surface integrals is probably best left until this point, for the discussion of these by purely analytical methods is not particularly easy. In this connection some attention should be given either here, or earlier, to the mathematical definitions of length, area and volume.

A fuller discussion of the theory of series, the theory of functions of a complex variable, Fourier series, elliptic functions or whatever else it may be desirable to include in the course, will easily follow, but at this stage the order of development is not so important and may be varied a good deal. E. G. PHILLIPS.

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703. *Sextant and Quadrant.*

Of the two reflecting instruments, the quadrant and the sextant, one or the other is wrongly named. If *quadrant* be right, the other ought to be *trient*. If *sextant* be right, the other ought to be *octant*. The principle of reflection, as explained in all books, makes the angle to be measured *double* of its angle through which the moveable bar is moved from its zero; accordingly, an *octant* (45°) is enough to measure angles as far as a QUADRANT (90°): while a *sextant* (60°) is good for all angles up to a TRIENT (120°). Either the italics should have been taken in both cases, or the capitals in both. This makes confusion, even in works of science. Thus in Nichol's *Cyclopaedia of the Physical Sciences*, one of the very best of its kind in Europe, we find, first, that the arc of 60° is divided into 120°, meaning 120° semi-degrees, each of which gives a degree of observed angle. It is then said that the vernier shows 10″: no man alive can say whether this is 10″ for the arc, answering to 20″ of observed angle, or 5″ on the arc, answering to 10″ of observed angle: I believe the first is meant.—A. De Morgan (*N. and Q.* III. vi. 138).

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* *Pure Mathematics* (1925), p. 389.

MATHEMATICAL NOTES.

947. [A. 1. a ; V. 7.] *A Letter of De Moivre and a Theorem of Halley.*

Among the many minor mathematical serials published in England during the past two and a quarter centuries is

> "The | Mathematical Magazine : | and | Philosophical Repository. | Containing a variety of | Original Pieces, | in all parts of | Mathematical Science. | By | Mess. G. Witchell, T. Moss, &c. | Vol. I. | Printed for J. Wilkie, at the *Bible* in *St. Paul's Church-Yard* ; and sold | by all Booksellers in *Great-Britain* and *Ireland*. | MDCCLXI."

Only five numbers were published, April-August. On pages 41-43 of the June issue is the very interesting letter of De Moivre (1667-1754) reproduced below, with the correction of a few very obvious misprints. Can any reader state where Halley's "theorem" is to be found or otherwise assist in determining an approximate date for the letter ?

"*A curious Letter from* Mr. Abraham de Moivre, F.R.S. *to* Dr. Edmund Halley, F.R.S. *never before published.*

SIR,—I Believe I have hit upon the right method of demonstrating your theorem for finding the rate of interest in annuities, and of continuing it ; if there were occasion for it.

Let n be the number of years ; let s represent so many years purchase, as the annuity is worth present money ; x the rate of interest : The equation, expressing their relation, is

$$\frac{x^n-1}{x^n\times x-1}=s\text{ ; let }x\text{ be made}=1+z\text{, therefore }\frac{\overline{1+z}|^n-1}{\overline{1+z}|^n\times z}=s\text{ ;}$$

let both numerator and denominator of the first part of the equation, be reduced into infinite series, and we shall have

$$\frac{nz+n\times\frac{n-1}{2}zz+n\times\frac{n-1}{2}\times\frac{n-2}{3}z^3\text{ etc.}}{z+nzz+n\times\frac{n-1}{2}z^3+n\times\frac{n-1}{2}\times\frac{n-2}{3}z^4\text{ etc.}}=s.$$

And dividing both by numerator, and denominator, of the fraction by z, we shall have

$$\frac{n+n\times\frac{n-1}{2}z+n\times\frac{n-1}{2}\times\frac{n-2}{3}zz\text{ etc.}}{1+nz+n\times\frac{n-1}{2}zz\text{ etc.}}=s.$$

And taking the quote of the numerator, divided by the denominator, we shall have n multiplied into

$$\overline{1-\frac{n+1}{2}z+\frac{n+1}{2}\times\frac{n+2}{3}zz-\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}z^3\text{ etc.}}=s.$$

And because the co-efficient of the second term of this series is $-\frac{n+1}{2}$, I am very naturally directed to raise that series to the power $-\frac{2}{n+1}$, which is the reciprocal of it ; now this series thus raised, will be $1+z-\frac{n-1}{12}zz$ etc. which, having continued to the cube, I found the co-efficient of the cube to be 0. The co-efficient of the biquadrate, I did not pursue ; it seeming, at first view, to be something intricate. We have, therefore,

$$\overline{n\times 1+z-\frac{n-1}{12}zz}\Big|^{-\frac{n+1}{2}}=s,$$

and therefore $$\left|1+z-\frac{n-1}{12}zz\right|^{-\frac{n+1}{2}}=\frac{s}{n},$$

and consequently $$\left|1+z-\frac{n-1}{12}zz\right|^{\frac{n+1}{2}}=\frac{n}{s},$$

and $$1+z-\frac{n-1}{12}\times z^2=\left|\frac{n}{s}\right|^{\frac{2}{n+1}}$$

and $$z-\frac{n-1}{12}\times z^2=\left|\frac{n}{s}\right|^{\frac{2}{n+1}}-1.$$

Let $\left|\frac{n}{s}\right|^{\frac{2}{n+1}}-1$ be called y.

Therefore $z-\frac{n-1}{12}zz=y$.

Let $\frac{n-1}{12}$ be called c, and we shall have $z=\frac{1}{2c}+\sqrt{\frac{1}{4cc}-\frac{y}{c}}$; and, consequently, the rate of interest $=1+\frac{1}{2c}+\sqrt{\frac{1}{4cc}-\frac{y}{c}}$. Now make, in this theorem, $\frac{1}{2c}=b$; that is, $\frac{6}{n-1}=b$. And we shall have the rate of interest $=1+b+\sqrt{bb-2by}$, which is your very theorem.

Now I take this to be extremely near the truth, when n is small; for it goes so far as the cube exactly, since the co-efficient of the cube is 0; and it participates of the biquadrate, and all the other powers.

I am, Your very humble Servant, A. DE MOIVRE."

Brown University, May 6, 1929. R. C. ARCHIBALD.

948. [V. 1. a.] *The Idea of Function and Manifold.*

Do not the remarks on functions by D. N. L. in the *Gazette*, May 1929, p. 463, show once more the existence of a kind of geological fault running through the *Gazette*; a want of fit between questions and their answers, or between demand and supply?

On the one hand we have some members of the Association who seem to think that the right foundations for school mathematics at the present day are to be found in the ideas of function and of manifold rather than in Greek mathematics. Their attitude resembles that of the bard who sang, "These heroes of antiquity ne'er saw a cannon ball." Our students live, they say, in a world very different from that of Euclid; a world of Meccano, of wireless outfits, of motor dashboards, of linkages between pointer readings. As Euclid used the familiar knowledge of his day on which to build, why, they ask, should not we use the equally familiar, but different knowledge of our day on which to build? Do we not honour the Greek mathematicians more by following their methods than by using their obsolete foundations?

On the other hand there are the experts. They show great skill in putting new wine into old bottles, but is that quite what is wanted? Will those who are fascinated by the ideas of function and of manifold, and who think that daily life abounds in simple illustrations of them, be satisfied by finding what they consider the foundations of the building located somewhere in the attics? They want to *begin* with those ideas, and it is the word "begin" which causes

the trouble. Not unnaturally, some of those who have spent much of their lives in introducing ideas in a certain order are inclined to think that to suggest a radically different order is hardly a sign of intelligence.

Teachers of physics and mechanics won the right to *begin* with modern views only after a long and arduous struggle against the classicists, and perhaps in the case of mathematics this battle has still to be fought. At any rate, there seems to be sometimes what I have called a geological fault perceptible in the *Gazette*: a want of entire sympathy between those who supply most of the talk, and those who fill the humbler *rôle* of a "noise of a crowd without."

From the older standpoint, to begin mathematics with functions and manifolds may seem like beginning with a muddled and difficult kind of Cartesian Geometry, and therefore quite unpractical. The objectors perhaps fail to grasp that it is possible to *begin* with definitions of words like "point," "straight line," "right angle," which are radically different from those of Euclid, but which lend themselves to practical illustration quite as much as Euclid's.

Our guide to the new definitions is, however, perfectly simple and obvious. For describing dependence or linkage between variables, we adopt the language which *would be used if* the dependence were to be graphed on squared paper, but we apply it *irrespective of whether it is graphed or not.* We disregard the historical origin of our words, and teach them without assuming any knowledge of graphs. For example, the association of two values becomes a "point" after graphing, and so we call it a "point" before graphing. Variables are "straight" after plotting along the axes, and therefore we call them "straight" by definition before plotting. A function is represented by a "line" after graphing, and therefore we call the dependence of the variables a "line," whether graphed or not. Two variables are "at right angles" after graphing, and so we define all variables to be "at right angles" whether graphed or not.

Certainly the propriety of using all these words could be quibbled about almost endlessly. To begin with, some may object (with more or less reason) to calling the subject itself "Geometry." In truth it is a *descendant* from the old geometry and algebra and might be described by a combined word such as Algeometry. If we were to invent brand new words, they would probably be a bigger nuisance than the old, and that is some justification for retaining the old, though one must admit that the retention gives some scope for confusion.

The two systems of definition agree when we are dealing with axes of co-ordinates which are "straight" and "at right angles" in Euclid's meaning. Thus it may be said that the older system is obtained from the newer by narrowing down the meaning of the newer definitions, by a judicious admixture of physics, until they are still suitable for the little region of applied mathematics studied in Euclid, but suitable for no other. Similarly, the passage back from the old to the new may be described as a reversal of the above process, or a judicious removal of the physics from Euclid. By this removal the stretched string, the footrule, and the setsquare become no longer necessary, their part being filled by definitions. In the newer treatment the variables or dimensions (for the two words mean the same) are all on the same footing, and it matters not whether they are read from tachometers, or clocks, or voltmeters, or pressure gauges.

Such are the thoughts which should be recalled by the sight of the dashboard of a car, or the switchboard of an electric plant. Thus, from this standpoint, the most important apparatus for teaching mathematics is not the ruler or compass, but some kind of artificial dashboard. (Youth may prefer to call it a jamboree of variables, or a mass meeting of clocks.) What is wanted is some kind of super-Meccano which will illustrate the real nature of the "bricks" and of the "building" used in mathematics. For the child's box of bricks, so often referred to, is a misleading analogy, the true "bricks" being linkages or functions between variables. And the mode of "building" used by the

mathematician could be nicely illustrated by the child's box only if it were our habit to build houses by putting together say a garage and a church, and then eliminating the partition wall.

Peterborough. T. C. J. ELLIOTT.

949. [I. 1.] *Noughts and Crosses.*

In a recent examination a question was set, which involved finding with the use of logarithms the values of

$$\text{(i) } 2^{64}-1; \qquad \text{(ii) } 2-2\left(1-\frac{1}{2^{64}}\right).$$

Very few seemed to understand that while it is correct (though perhaps not elegant) to give the value of the 2nd term in (ii) as

1·999,999,999,999,999,999,8911,

a similar string of nines in the answer to (i) is completely wrong, *not* because 1 is negligible in comparison with 2^{64}, but because when we write 2^{64} as 1837 followed by sixteen noughts these noughts are not zeros but unknowns.

It would hardly be worth while I suppose to turn some of our noughts into crosses and write 1837*x*,*xxx*,*xxx*,*xxx*,*xxx*,*xxx*, when we have available index notations which are so much more convenient, but perhaps some of our less bright pupils might find the meaning of the *x* less an unknown.

C. O. TUCKEY.

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704. N.L.T. reported that he had lately asked an "old inhabitant" his age; and he answered ... "Why, Sir, I belong to the sevens; born in the three sevens (1777), I must this year (1854) of course confess to the two sevens (77)." Another century must elapse before this reply can be given, after the year which has just expired. (*N. and Q.* I. xi. 61.)

W.T.M. of Hong Kong appealed on behalf of strict correctness to be allowed to remark that this reply, *ipsissimis verbis*, can never be given again. "An analogous reply might be given in the year 1976: I was born in the three eights, and confess to the two eights." It is singular that with each figure the lapse is of 122 years precisely.

"M" 's characteristic retort is as follows:

Though W.T.M. dates from the end of the world in space, I cannot permit him to know so much about its end in time, as to affirm that the reply can *never* again be given. A man born A.D. 2777 may surely make it in 2854. And farther, there is nothing singular in the interval being 122 years; 111 and 11 make 122.

Of my own age I may say something which will not be predicable at equal intervals. I was *x* years old in the year of grace $x \times x$. I will say so much as, that I do not mean I was 6 years old in A.D. 36 nor 7 in A.D. 49. I dare say Professor De Morgan, or some of your mathematical correspondents, will be able to find my age.—M. (*N. and Q.* I. xii. p. 94.)

[This reply is credited in the Index to the *M.* who contributed other replies that were certainly from the pen of De Morgan. Apart from that there is other conclusive evidence, for in 1849 De Morgan was 43 years old.]

705. All writers who have hitherto written *Supplements* to Euclid have used Euclid's methods. There is a trisection of the angle, made by unbending an arc into a straight line: Euclid would have died of this Supplement.—Review of a *New Supplement to Euclid's Geometry*, Athenaeum, 1841, p. 226.

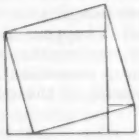
706. The Cambridge Problems are unquestionably a more curious and ample collection of mathematical conundrums than can be elsewhere found. Containing a great deal that is very trashy, and much that is merely whimsical, there is withal a considerable residue of sterling science and ingenuity.—*Westminster Review*, July 1825, page 142. [Per Mr. V. Naylor.]

REVIEWS.

Mathematics for Students of Technology: Junior Course. By L. B. BENNY. Pp. 267, index and answers. 4s. net. Also in two parts. 2s. 3d. each. 1929. (Oxford Univ. Press.)

This is intended for a two-year course in elementary mathematics; and more especially as an introduction to the "Senior Course" by the same author. On the whole this book should be found admirably fitted for its purpose.

Among the praiseworthy features are: (i) the placing of the decimal point in all cases of multiplication and division by the use of a preliminary rough check, and the use of "algebraic" multiplication: (ii) the introduction of algebraical symbols through mensuration, as generalized arithmetic; (iii) the very careful introduction of trigonometrical notation in a manner likely to interest as well as instruct the student; (iv) the inclusion of a good deal more pure geometry than is usual in text-books of this kind, together with the insistence on the need for rigorous proof. In connection with the last subject, however, it is hard to see how the proof given for Pythagoras' Theorem meets



the author's hope that it should be easier than the ordinary proof given by Euclid. If ease of proof alone is considered, it is suggested that the author, in subsequent editions, should give what is generally considered to be Pythagoras' own proof, or the proof of his School, that which uses the figure of Euclid VI, 8, a right-angled triangle with the perpendicular on the hypotenuse; while as a preliminary dissection-proof in the first part, he should use the diagram given above, for which a short proof can also be added, if no insistence is made on all the details being written out.

The author is hardly, in my opinion, so happy in his introduction of negative numbers. The heading, Negative Quantities, confuses the logical point that negative numbers are *invented* to make $a-b$ intelligible in all cases; and that this is done by extending the scale of natural numbers backwards, the numbers *afterwards being applied* to quantities. This is not only the only correct way, but it would be the logical outcome of the author's introduction of algebra as generalized arithmetic. In the Rules of Signs, he tries to explain what is a *definition* of the operation of multiplication by these new numbers; the only possible "explanation" being that the rule is necessary in order that $(a-b)(c-d)$ should always be equal to $ac-ad-bc+bd$. In the article and examples on brackets, the signs of negative number and of subtraction are confused. All these subjects logically follow the first few sections of the author's Chap. X, and would with advantage be deferred till then.

Logarithms are left until a comparatively late stage. The author states in his preface that he does not see how a student is to understand them if they are introduced earlier. It is hard to see how a student, of the sort this book is intended for at any rate, is to understand them when they are introduced as indices of powers of 10, without much work on approximate values done in a highly theoretical manner, such work as such students could not afford time for. An irrational number, such as $\sqrt{2}$, even when it is presented in the author's clear manner, as being defined by the approximation $1{\cdot}41<\sqrt{2}<1{\cdot}42$ will be hard enough for most technical students; while, the logarithm of 2, defined similarly by $10^{\cdot 3010}<2<10^{\cdot 3011}$, where the defining approximations are themselves *irrational*, provides a difficulty to the majority of First Year Honours students of Universities. If logarithms are introduced as "tables of addition for multiplication," the "rule of thumb" that the author despises, but the historical method in which they were not powers of 10, there is opportunity for good logical and instructive teaching in bringing out all the rules for

logarithms from the three defining equations: $\log 10 = 1$, $\log 1 = 0$, and $\log ab = \log a + \log b$. Especially does this way bring out the need for writing numbers in "Standard Form." The foolish use of logarithms, which the author rightly deprecates, for calculations that could be almost performed mentally, a really vicious habit, could easily be stopped, especially by teachers using this book carefully, by insisting on the approximate nature of logarithms.

J. M. C.

The Universe Around Us. By Sir J. H. Jeans. Pp. 352. 12s. 6d. 1929. (Cambridge University Press.)

Astronomy and Cosmogony, by the same author, appeared about a year ago, and was reviewed in these pages; that work, while containing much matter that all could understand, was of an avowedly mathematical character, and many of its pages looked somewhat repellent to the uninitiated. The present work is intended for a wider audience, and eliminates all mathematical formulae, though it contains the results originally reached by such formulae, but here expressed in plain and attractive language; for example the author discards the technical term *mass* in favour of the more familiar *weight*.

The chapter "Exploring the Sky" gives a sketch of our knowledge of the planetary and stellar systems, leading on to the recent determinations of the distances of the spiral nebulae, which in turn suggest some discussion on the curvature and radius of space. The author notes that the measures made at Mount Wilson of the velocities of recession of the more distant spirals favour de Sitter's estimate of the radius of space as some 2,000 million light-years, which is fourteen times the estimated distance of the furthest visible spiral; the estimate adopted by Jeans in his earlier book was some forty times as great. He brings forward the motions of the Magellanic Clouds and the two nearest spirals as opposed to de Sitter's hypothesis; it has, however, been suggested that these latter motions can be explained by the general rotational movement of our own universe.

Chapter II takes us abruptly from the extremely great to the extremely small, being a sketch of the new physics of the atom, which dates from the discovery of radio-activity in 1896; there are some interesting plates, taken by the method of Prof. C. T. R. Wilson, in which the behaviour of the minute flying particles, atoms and electrons, can be readily traced; the Quantum Theory, and the connection of atomic structure with spectral lines are described in the simplest terms that these recondite subjects permit; I may refer to the amusing explanation of the quantum theory, which is illustrated by describing the plight of a millionaire who cannot extract a matchbox from an automatic machine, since he has no coppers.

Chapter III discusses the problem of the ages of the planets and stars; exact results cannot be obtained, but a number of lines of evidence are shown to tend to the conclusions that the planets were born a few thousands of millions of years ago, the stars a few millions of millions of years ago; thus the sun is so much older than the planets that it was in much the same physical condition as at present when they were born. The tidal theory, according to which the birth of the planets arose from the near approach of another star to the sun, and the consequent expulsion of a stream of matter from the sun which separated into the planets, is adopted in the book; if this is the true explanation, planetary families must be very rare; it is estimated that not more than one star in a hundred thousand has such a family.

Chapter IV deals with the subject that the author has specially made his own, that of the behaviour of gaseous masses under rotation; several beautiful photographs of nebulae are reproduced, and it is shown that many of the features that they present can be explained by the hypothesis that they are huge masses of gas in rotation; it is further shown that if all space were filled with diffuse nebulosity, the condensations that would form in it would contain thousands of millions of times the mass of the sun, so that they would be comparable with the masses found for the spiral nebulae.

We are then led into the deep interior of stars, where there is a temperature of millions of degrees; we are given the arguments for believing that the transformation of matter into radiant energy is going on in the depths; this

seems to be the only way in which we can conceive of stellar radiation being maintained for such enormous periods as millions of millions of years.

I cannot think it wise of the author to insert his extravagant estimate of the future duration of life on the earth as a million million years; he has certainly not made this estimate with anything like the same care that he has given to the rest of the book; it is a question not only of solar physics, but also of terrestrial physics, which are only dealt with here in a very superficial manner. It does not seem reasonable to picture the habitability of the earth as continuing till seas and rivers are "frozen packs of solid ice." How could vegetation subsist in these conditions? it is noted elsewhere in the book (p. 222) that after the lapse of one twentieth part of a million million years the day will be forty-seven times as long as it is now; it is doubtful whether the vegetable world would survive the keen frosts which would occur in such prolonged nights.

Further, he speaks of the mountains as reduced almost to plains by continued denudation; but the same action would put almost all the land under the ocean. As regards the sun, there is one possibility that he has not mentioned. The outbursts known as *Novae* or New stars are now known to occur several times every year; further, at least some of these stars, such as Nova Aquilae 1918, seem to have been in a similar spectral condition to our sun before the outburst; the very highest estimate of the number of stars in our universe is far below a million million; so that the odds are quite in favour of the sun being affected by such an outburst before the lapse of a million million years; from all these considerations, and many others of a similar kind, it will be seen that the prophecy is a decidedly rash one.

It is not necessary to take so pessimistic a view of the ultimate end of all creation as the author does. The Creator to whose action the beginning of the universe is ascribed (page 328) can save such parts of it as He desires from destruction. *Non omnis moriar* is the expression of a reasonable hope. Note may be made of a few passages in the book that might mislead some readers:

The paragraph about Copernicus is a little misleading; Copernicus was something of a theologian; he took his degree as doctor of canon law, and was appointed Canon of Frauenburg, though without taking orders; of his heliocentric theory we read in *Encycl. Brit.*, 14th edition, vol. 6, p. 400, that "Clement VII approved, and Cardinal Schönberg transmitted to the author a formal demand for full publication."

Page 18: Adams was eight years younger than Le Verrier, and did not become Professor till 1858.

Page 34: Eros cannot cross the sun's disc.

Page 45 (near foot): the ellipse does not cover "all the different shapes of orbit;" the parabola and hyperbola might have been mentioned for completeness.

Page 240: The sun's equator is inclined only some $5\frac{1}{2}$ degrees to the invariable plane of the solar system; this should hardly be described as "The quite different plane."

Page 244: Biela's and Taylor's comets were about an astronomical unit from the sun when they split in two; this is very far outside the Roche's limit of 2·45 radii of primary, given on p. 243.

Page 245: In speaking of very small bodies like the inner satellite of Jupiter it should be pointed out that for such bodies the force of cohesion is much stronger than the differential gravitational pull, and they may reach the surface of their primary without breaking up; if this were not so our terrestrial mountains would collapse.

Apart from the few points that I have mentioned as slight blemishes, the book is most interesting and stimulating, and its perusal can be strongly recommended to all who take an interest in astronomy, or indeed in any form of natural science.

A. C. D. CROMMELIN.

Vorlesungen über Differential- und Integral-rechnung. IIte Band, by R. COURANT. Pp. 360. 18.60 R.M. 1929. (J. Springer, Berlin.)

This book is the second volume following a volume previously reviewed, and considers functions of several variables. It also includes an introduction to the study of ordinary and simultaneous differential equations.

Although it is written as a text-book for University students it is much more carefully worded than is usual among such elementary works. Professor Courant has wisely chosen to restrict curves and boundaries of regions to those which are smooth in stretches (stückweise glatt), that is to say, divisible into a finite number of arcs on each of which (including its ends) the angle of slope is continuous. This restriction does not seriously curtail the applicability of the theorems while it makes possible elementary and rigorous proofs.

After an introduction on vector theory, the affine transformation and determinants (up to the third order) there is considered continuity in the plane and partial derivatives while applications are made to vector fields. Weierstrass' convergence principle and the Heine-Borel covering theorem are given and form a necessary foundation for any analysis, however elementary, which pretends to be thorough. There follows the usual topics in such a course, with a constant eye to applications in theoretical physics. Stokes' theorem and the Gauss and Green integrals are well handled. In differential equations unsymmetrical forced oscillations and planetary orbits are taken as applications.

While there are no sets of examples such as an English student would expect all the points are illustrated with care and there is a clever steering between the purely theoretical and the practical. The analysis is not so advanced as Goursat or de la Vallée Poussin, but it is more careful than in most English text-books in use and pays less attention to the merely complicated.

It is well printed and has a sufficiency of clear figures and a good index. Altogether a book which should be in the library of a University teacher and even in a school library for scholarship candidates.

Zehn Vorlesungen über die Grundlegung der Mengenlehre. By A. FRAENKEL. Pp. 180. 8.00 R.M. 1929. (Wissenschaft und Hypothese XXXI. B. G. Teubner, Leipzig.)

Einleitung in die Mengenlehre. By A. FRAENKEL. Dritte umgearbeitete und stark erweiterte Auflage. Pp. 424. 24.00 R.M. 1929. (J. Springer, Leipzig.)

At the present moment Mathematics is almost as insecure in its foundations as is Physics. There are several schools of opinion typified by Russell, Brouwer and Hilbert respectively. All of these deserve a sympathetic hearing but none quite satisfy us.

The reviewer's personal opinion is that the mathematician too often attempts to attain a security which no other sciences except Physics have possessed and which even that science has now been compelled to give up. Mathematics is a doing, not a knowing. Let us remember the fate of the great reptiles and put our trust, not in armour, but in the most agile and persistent intelligence.

The two books before us discuss the difficulties which have arisen in the theory of classes. The first, smaller, book is composed of a set of lectures delivered to an audience not composed of those familiar with the technique of the subject; the second is intended as a general introductory text-book and is rightly stated to be a completely revised and enlarged edition of a book already known to many. Both of the books before us cover the same ground in the same order and with very much the same arguments. The second is more thorough in details and gives a valuable bibliography. But while the shorter one might well be recommended to follow immediately after Russell's *Introduction to Mathematical Philosophy*, the longer one disappoints by a stream of verbiage and by refusing to enter the fruitful lands of the theory of sets of points and of abstract geometries—fields such as those described by Hausdorff and Fréchet. One has the impression that Professor Fraenkel, in spite of his long and clear expositions, does not come to close enough grips with any of the various developments which are criticized. In all cases he confines himself to the initial bodies of assumptions. This is particularly noticeable in the larger book in which there is ample room for positive analysis. Thus the reader does not learn how far, or in what particular suburbs, Brouwer and Weyl have succeeded in rebuilding the city they have so zealously destroyed. Again the considerable constructive abilities of Russell and White-

head are not impressed upon the reader; their doctrine of hierarchical types is criticized on the ground that the upper bound of a class of real numbers must be of a higher type of real numbers whereas it is known that this upper bound can be defined as a class of rationals (cf. Hobson or Knopp). The doctrine of propositional functions is rejected on similarly slender grounds. Professor Fraenkel seems to prefer the apparent solidity of classes "stretched before him." In this he falls into the usual trap of regarding mathematical objects as if they were billiard balls—a trap from which Russell saves us.

The development divides itself into three parts. In the first is given the classical Cantor theory while the third is devoted to Zermelo's system of axioms—a system which has much to commend it, especially as a sufficient basis for the workaday mathematician. But it suffers from an impossible distinction between classes and properties and is too shallow after the depths to which the reader has previously been led.

In this system there arises the problem of the significance of an attribute. This problem of significance has not received from logicians the attention it deserves. It includes not only that of hierarchical types, but also such attributions as "apples are courageous."

The smaller book can be recommended to those who wish to know something of the fight going on, and it is to be hoped that it will help to preserve us from the complacent adoration of the Truth of Mathematics. P. J. D.

Algebraic Geometry and Theta Functions. By A. B. COBLE. Pp. iv + 282. 3$. 1929. (American Mathematical Society.)

There is always a charm in finding the connection between subjects that one has approached before from opposite sides. Mr. Coble's lectures present the common ideas that underlie certain geometric configurations and certain properties of theta functions. The functions of genus 2 are related to the Weddle surface, those of genus 3 to sets of seven points in a plane and eight in space, and those of genus 4 to the tritangent planes of a space sextic. Novelty is claimed for a dozen or so results; there is a full bibliography.

Lectures seldom make an attractive book, either for reading or reference. In order to be intelligible, more than a third of the book is given to two preliminary chapters on the two sides of the theme; in order to be compact, the notation is condensed so as to make the rest very stiff reading. But those who submit to the discipline will have their reward. H. P. H.

Statistical Mechanics. By R. H. FOWLER. Pp. 570. 35s. net. 1929. (Cam. Univ. Press.)

This treatise on "The Theory of the Properties of Matter in Equilibrium" is almost encyclopaedic in content and contains a mass of material not to be found elsewhere except in original contributions to scientific periodicals. But the chief value of this monumental work lies neither in the wide extent nor in the satisfying detail of its treatment of equilibrium theory, but rather in the novel and systematic manner in which the principles of the theory are elaborated.

The equilibrium theory of statistical mechanics has been approached in the past both from the *à posteriori* and the *à priori* aspects. Both methods of approach had as their common object the determination of the "normal properties" of a conservative assembly of dynamical systems: but, whereas the first method regarded normal properties as those actually established by a specified mechanism of collisions between the interacting systems, the second method was based on the definition of the "normal state" as that possessing the greatest number of weighted complexions, *i.e.* the maximum statistical probability. The outstanding feature of the work under review is the manner in which (I) it develops the *à priori* method, and (II) modifies both this and the *à posteriori* method so as to be in harmony with the requirements of wave mechanics.

I. The development of the *à priori* argument is determined by three principles which relate respectively to the definitions of (1) the statistical assembly, (2) its normal state, (3) its thermodynamic properties.

(1) The statistical assembly is taken to be an assembly not of classical systems but of quantised systems. Throughout the greater part of the book the older quantum theory is used, the minor modifications required by the later developments being noted without proof and the major modifications relegated to the last chapter. The distribution laws for classical systems are derived by a limiting process from the similar laws for quantised systems. Only discrete energy values are permissible in quantised systems and there is a corresponding gain in the precision and elegance with which the state of the assembly can receive mathematical formulation. To each state of a non-degenerate system is assigned unit weight, and degenerate systems are treated on the assumption that weights are adiabatic invariants.

(2) Fowler and Darwin define the normal properties of the assembly, not as the "most probable" properties, but as the average properties possessed by all accessible assemblies, *i.e.* all assemblies transformable into one another. The older definition of the most probable state was made more palatable by a proof that it was "infinitely probable" compared with any other state: similar corroborative evidence is forthcoming in support of the newer definition of the average state. It is shewn that if P_0 is the average value of P, then the proportional fluctuation of P, viz.

$$\{\text{average of } (P-P_0)^2\}^{\frac{1}{2}} \div \{\text{average of } P\},$$

is of the order of $P_0^{-\frac{1}{2}}$ and is therefore insignificant when P_0 is large.

In the calculation of the average properties the number of weighted complexions is not estimated by an appeal to Stirling's Theorem but by the identification of the number required with the coefficient of a power of z in the expansion of a "partition function" $f(z)$. This coefficient is expressed as a contour integral and the integral evaluated by the method of steepest descents.

(3) This method requires the determination of the position of the maximum of the modulus of the integrand, and this is expressed by a number of parameters which are interpreted by the second law of thermodynamics as the temperature and partial potentials of the constituents of the assembly. At the same time there is obtained an expression for the entropy S and a proof of Boltzmann's relation, $S = k \log W$.

The foregoing principles are applied, first to permanent assemblies (perfect gases, radiation and crystals), then to assemblies which suffer dissociation and evaporation (Nernst's heat theorem, thermionic phenomena), and to assemblies with systems not fully independent (imperfect gases, dilute solutions). Further tentative applications are also made to the magnetic and dielectric phenomena of gases and the high temperature phenomena of stellar interiors and exteriors.

II. Fresh light is also cast upon the *à posteriori* method by regarding Boltzmann's H-theorem as the proof that the preservation of equilibrium requires detailed balancing. The new statistical mechanics of Bose-Einstein and of Fermi-Dirac may then be regarded as the consequences of making the probability of collision depend not only on the incident elements but also on the products of the collision.

As regards the *à priori* method the major change due to the newer forms of quantum theory depends upon the limitation of the wave-functions of an assembly to symmetrical or antisymmetrical forms. This has the effect of reducing the number of accessible states of the assembly and thus altering the distribution laws. The most important application is to Sommerfeld's new electron theory of metals.

It must be stated in conclusion that in places the argument is extremely compressed and a reference to Fowler's original papers is almost necessary for its comprehension. Throughout the work a good knowledge of the elements of the subject is presupposed and, in particular, the last chapter requires a working acquaintance with wave-mechanics. The value of the work would be enormously enhanced by the publication of a small introductory tract to serve as a guide-book to the fascinating land partly seen, partly surmised, from this latest monument of Cambridge scholarship. G. TEMPLE.

Dynamics: A Text-Book for the use of the Higher Divisions in Schools and for First Year Students at the Universities. By A. J. RAMSEY. Pp. xii + 259. 10s. 6d. net. 1929. (Cambridge University Press.)

This book appears to the reviewer to be the book which has been so long awaited, a text-book on Dynamics really suitable for Mathematical Specialists.

Since Loney's *Elementary Dynamics* there has been no similar book likely to prove so generally useful.

Though it begins at the beginning it is not a beginner's book, but is suitable for those who have worked through one of the elementary books which aim at the standard of additionals in School Certificate and who have also done some Calculus.

It should then suffice for the subject for the rest of their school career.

The methods of the Calculus are freely used and the amount of ground covered is somewhat larger (but not very much so) than the present course for entrance scholarship examinations at the University.

As the author explains in his preface the Mechanics of the Mathematical Tripos is an expanding subject; for instance "motion under gravity" is now understood to allow "in a resisting medium," and he regards it as inevitable that the effect of this kind of growth in the content of Elementary Dynamics will extend to the schools.

The result of this is that there is a refreshing novelty in the appearance of some of his chapters, or rather in certain sections of them devoted to matter which in school teaching is new.

Thus the chapter on "Polar Coordinates, Orbits," without appearing specially difficult, includes the treatment of a variety of orbits described, under the influence of force directed towards a fixed point and a comparison of the effect of Newton's law of gravitation with that of Einstein's law, showing that the latter causes an "advance of perihelion."

Other attractive chapters are Chap. XVI, Miscellaneous Problems, and Chap. XVII, Small Oscillations.

Each chapter concludes with a set of examples, mostly 20 to 30 in number, mainly taken from Tripos and Scholarship papers.

The reviewer would urge all keen teachers of Dynamics to mathematical specialists to examine this book, if he did not expect that most of them have already adopted it.

C. O. TUCKEY.

The Principles of Mechanics. By H. C. PLUMMER. Pp. xii + 307. 15s. net. 1929. (G. Bell & Sons.)

This book professes to be a first stage book for those "with only a slight equipment of mathematical knowledge," "suitable for those who will ultimately go further." The reader is "warned that the beginnings of the Calculus will still lie before him."

But it is difficult to envisage the type of beginner capable of tackling the book, especially if ignorant of Calculus.

The curvature of a roulette is dealt with in the first chapter—this does not appear to be an unfair example of the standard—and in general the notation of the Calculus (including the unusual $\iint \ddot{s}\, dt$) is freely used.

It is true that the notation is explained as it is introduced but it needs more than careful explanation by the writer, it needs much practice with examples by the reader before the beginner can hope to grasp such a statement as (p. 86):

$$\rho = (\dot{r}^2 + r\dot{\theta}^2)^{\frac{3}{2}} \div \{r\ddot{r}\dot{\theta} - \dot{r}(2\dot{r}\dot{\theta} + r\ddot{\theta}) - r^2\dot{\theta}^3\}.$$

The general impression made by the style of presentation is that the book is suitable for the Mathematical Scholarship candidate in his second year of specialisation in Mathematics, or perhaps for the first year of the University course.

This ensures that the reader will not be a beginner in Mechanics, except in very rare cases.

In fact, it seems better to regard the book as a revision course for those who have already studied one of those books in which "a certain range of elemen-

tary problems is contemplated without much regard for the need of a coherent foundation."

From this point of view there is much to commend in it.

The bookwork is set out with considerable care; the worked-out examples are judiciously chosen and their solutions are given to a suitable degree of detail; and each of the nine chapters concludes with a set of 25 to 30 examples of "scholarship standard," in the usual sense of that somewhat elastic phrase.

The arrangement of the chapters is as follows:

The first three deal with Kinematics, Chaps. IV, V with Dynamics of Translation, Chaps. VI, VII with Statics, Chap. VIII with Dynamics of a Rigid Body in two dimensions and Chap. IX with Elasticity.

In common with many writers the author uses g either for the acceleration due to gravity or for its numerical value. This is very convenient, but so careful a writer might have been expected to say beforehand that he proposed to do so.

The author takes the view that Newton's second law is merely a definition of force. He gives an illustration in which pressure of gas balances either a weight or the pressure of a hand and explains that while we know the pressure of the gas as rate of change of momentum, we really know nothing of the opposing forces in spite of their apparent familiarity.

It is the reviewer's opinion that if Newton's second law were merely a definition, then these three celebrated laws would have proved much less adequate as a foundation of Dynamics than they have proved, and that if Newton had intended his second law merely as a definition of "*vis impressa*," his knowledge of Latin was quite sufficient to enable him to make his intention clear—but probably this is personal prejudice.

In any case the discussion given of Newton's laws is interesting and so is the discussion on the limitations of the principle of Conservation of Energy in Dynamics.

In the Statics an unusual arrangement is adopted.

The parallelogram law gives the rule for the balancing of two couples, and from this all needed results as to parallel forces are derived. On the whole these chapters seem less interesting than those on Dynamics.

The chapter on Rigid Dynamics is largely concerned with moments of inertia and with pendulums, the latter being dealt with at some length.

The chapter on Elasticity has nothing to do with collisions, but deals with elastic strings, plates under stress, bending of rods and such like.

The book gives too short a treatment to be suitable as sole book at any stage, *e.g.* a chapter of 30 pages with 30 examples has to suffice for the whole of Statics, except parallel forces and centre of gravity.

But the careful discussions of the principles make it an interesting revision book suitable for able pupils in their last year or two at school or first year at the University.

C. O. TUCKEY.

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707. *Eureka.* Liverpool, 9th October, 1855.

Sir,—I do not know whether I am in place in asking for a nook in your valuable journal for the squaring of the circle. I conceive it to be simply this:

If you take a silver wire twelve inches and a quarter long, the quarter being allowed to unite the two ends, you have a circular wire exactly twelve inches; and if this wire is made to form its third square, each of its sides will be equal to three inches, and its area equal to nine square inches. Now, if the same wire is allowed to assume the true circle, it is evident that the area of the circle will be the same as it was in the square. For instance, if a wall be built around a city, and it is found to be twelve miles long, the area of that city is nine square miles. Therefore the square of any circle is equal to three-fourths of the length of its own circumference.—R. D. S. (*N. and Q.* I. xii. 363). Quoted from a letter to the *Spectator*, Nov. 3, 1855.

THE LIBRARY.

160 Castle Hill, Reading.

Proceedings of the London Mathematical Society.

Thanks to the combined generosity of several donors, followed by exchanges of duplicate volumes, the Library has now both the original series complete and the second series complete to vol. 27, part 4. Subsequent parts and volumes would be very welcome. Moreover, duplicates, especially of early volumes, can always be exchanged to advantage.

Nieuw Archief voor Wiskunde.

Wiskundige Opgaven met de Oplossingen.

These periodicals, with which the *Gazette* has exchanges, have now been secured complete from their beginnings. They are the products of a Society at Amsterdam that bears the title *Een onvermoeide arbeid komt alles te boven*, and their history is so long that a bibliographical epitome must suffice.

(1) Archief, uitgegeven door het Wiskundig Genootschap, onder de Zinspreuk: *Een onvermoeide . . .*, te Amsterdam.

Vols. 1-3: 1856-9, 1860-6, 1870-4.

Nieuw Archief voor Wiskunde.

Vols. 1-20: 1875-1893; two vols. dated 1882, none 1887, two 1888.

Nieuw Archief voor Wiskunde. Uitgegeven door het Wiskundig Genootschap te Amsterdam.

Series 2. Vols. 1→: 1895→.
The current volume is 16.

(2) Verzameling van Wiskunstige Opgaven, door de Leden van het Wiskundig Genootschap, ter spreuke voerende: *Een onvermoeide . . .*, elkander tot onderlinge oefening opgegeven.

Vols. 1-2: 1850, 1854.

Wiskunstige Opgaven met hare Ontbindingen, uitgegeven door de Leden van het Wiskundig Genootschap: *Een onvermoeide . . . . . .*

Four volumes without serial numbers, the titles continuing: van 1855-1859, van 1860-1865, van 1866-1870, van 1870-1874.

Wiskundige Opgaven met de Oplossingen, door de Leden van het Wiskundig Genootschap, ter spreuke voerende: *Een onvermoeide . . .*

New Series, Vols. 1→: 1875-1881→.
The words "Nieuwe Reeks" were dropped from the title page at Vol. 12; the current volume is 14.

From Miss **D. M. Lea,** valuable runs of the *Gazette* and of the *Proceedings* of the London Mathematical Society, together with school books by J. Casey and H. St. J. Hunter, and the following:

P. Barlow	Tables {2 (1840) ster.} - - - - - - -	1882
	With De Morgan's preface, but without his list of the mistakes in the original edition of 1814.	
F. Callet	Tables Portatives de Logarithmes {(1795) ster.} -	1837
C. Hutton	Mathematical and Philosophical Dictionary {2} (2 vols.)	1815
	"To be considered almost as an entire new work."	
J. Napier	Wonderful Canon of Logarithms - - - -	1857
	"Retranslated from the Latin text, and enlarged with a table of hyperbolic logarithms to all numbers from 1 to 1201," by H. Filipowski. It is perhaps because of its unpretentious form that this edition was reported to Macdonald from only three libraries.	
W. Ritchie	Principles of the Calculus {2: J. A. Spencer} - -	1847
	De Morgan refers to this book in favourable terms in the *Budget of Paradoxes*.	

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

SIR,—When professional mathematicians differ, the smaller fry of school teachers and the like must generally be content to await the issue ; but when the difference bears on points which touch or lie close to their work, teachers may perhaps be allowed to ask questions.

Dr. Robb's article on "Partial Failure of Euclid I, 4" in your July number is a case in point.

He presents the matter as one arising out of Time-space theory, but it appears to be in essence a mere question of geometry—a geometry, if I understand rightly, which was developed long before it found the Time-space application which has made it important.

The received doctrine appears to be that in this geometry there are two classes of lines, distinguished from one another by the fact that the lengths of those of one class, it does not matter which, contain the factor i. The two classes are separated by a pair of special lines, called I believe "isotropic," directed to I, J the circular points at infinity which in this geometry are real.

These are, as I understand, the lines which Dr. Robb calls "optical lines," and it is in regard to these that my perplexity arises.

The received doctrine says that the lengths of all lines lying along isotropic lines are zero, as indeed seems natural, seeing they belong to both the classes, or to neither.

But Dr. Robb asserts that such lengths are not zero, though he gives no reason for his assertion.

Again, referring to the figure on p. 475, I understand that the received doctrine regards the right-angled triangles OQP, $OQ'P'$ as congruent, their third sides OP, OP' being of zero length. Dr. Robb says they are not.

As to the triangles RQP, $R'Q'P'$, which again I should have understood to be congruent, Dr. Robb says on p. 476 that they are not—that QP, $Q'P'$ are in the ratio of x_0 to x_0' ; while on p. 475 when he was considering the same lines as elements of OQP, $OQ'P'$ he apparently regards them as equal.

It is very perplexing ! May we have some elucidation ?—I am, Sir, Your obedient servant,

25th July, 1929. W. C. FLETCHER.

SIR,—I am sorry that Mr. Fletcher should have found difficulties in my article, but I hope that he may not find these insuperable.

The substance of my statement was that : to take lengths measured along optical lines as being zero, gives a wrong idea of what actually occurs : since lengths can be compared along the same or along parallel optical lines in just the same way as they can along the same or along parallel lines of either of the other types.

By means of a simple construction of parallelograms on the same base and between the same parallels, it is possible in this, as in ordinary Euclidean geometry, to divide any given line into equal parts and, if we select a unit segment, we may construct any multiple of that segment.

This construction applies to optical lines equally well with either of the other types of line and so, to this extent, an optical line has a property similar to the latter, and this property enables us to make use of Euclid's criterion of proportion. The difficulty arises when we try to compare lengths along lines which are not co-directional. In the case of any two inertia lines or any two separation lines, it is possible to give constructions for doing so ; and these will be found in my *Theory of Time and Space* ; but in the case of optical lines there is no analogous construction : so that the only true congruence in the case of optical lines is co-directional.

As regards Mr. Fletcher's reference to the triangles OQP and $OQ'P'$; it is evident that the points O, P and P' lie in one line and, as the figure is drawn, P is between O and P'.

Thus OP is a part of OP' and so, unless one is prepared to give up the axiom that "the whole is greater than its part," we must conclude that OP' is greater than OP.

Mr. Fletcher's final difficulty arises, so far as I can see, from his confusing the pair of triangles ORP and $OR'P'$ with the pair QRP and $Q'R'P'$. It is the former of these pairs of triangles, (not the latter), which have each got two sides equal to b and c, while the included hyperbolic angles are equal to $\log b/c$.

The third sides of these triangles, that is to say OP and OP', are in the ratio of the corresponding values of x_0, and I think that, if Mr. Fletcher will examine my paper again, he will find that this is what I assert.

1st August, 1929. ALFRED A. ROBB.

SIR,—Writing away from home I have not Dr. Robb's article at hand, but, of course, I accept his comment on the last paragraph of my letter and can only express my regret for my careless misreading.

But this does not touch the essential point, viz. that OP, OP' are as I understand of zero length, and as to this—or rather the contradiction between Dr. Robb's and the accepted doctrine—his rejoinder leaves me unenlightened.

8th August, 1929. W. C. FLETCHER.

ERRATA.

P. 430, l. 11. *For* to Marlborough *read* of Marlborough.

P. 464, Gleaning 679, l. 1. *For* brining *read* bringing.

P. 510, l. 2 up. *For* Lynnersley *read* Kynnersley.

P. 532, Gleaning 687, l. 5. *For* $\frac{333}{166}$ *read* $\frac{333}{106}$.

P. 533, ,, l. 17. *For* "eatablished" *read* "established."

NOTICE.

The Editor will be glad to receive *short* passages to add to the collection of *Gleanings* in the Gazette.

Will Mr. Arnold J. W. Keppel be kind enough to send his address?

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MATHEMATICAL ASSOCIATION.

LONDON BRANCH.

THE Summer Meeting of the London Branch is of the nature of an excursion. It has been found that a visit to a Public School of ancient foundation has a special fascination for members. To see something of the activities of the school in classroom and playing field, to make an acquaintance with its traditions and its historical associations, to see in its memorials a record of the grateful loyalty of old boys, are especially interesting. On 8th June Mr. and Mrs. Hope-Jones devoted an afternoon to entertaining some fifty or more members at Eton. Thanks to their kindness, the visitors spent a most enjoyable afternoon, and thanks to the excellence of their stage-management every moment of the time was fully used.

A MEETING was held on Saturday, October 5, at Bedford College, with Mr. Katz in the Chair. There were 90 present.

Dr. P. B. Ballard gave a talk on Modern Methods. Thirty years ago the teaching of elementary arithmetic was purely mechanical. There followed a period in which mathematical instruction was based on the belief that it developed intelligence. Experiment made it very doubtful whether mathematical instruction made any special contribution to that development. Its purpose was to impart mathematical notions with economy of effort. It should present ideas and form habits, providing for conscious learning and automatic use. Understanding should precede mechanical repetition and in the latter process it was inadvisable to require logical thought. There was no antithesis between intelligence and habit; a proper adjustment between the two was required. It was not difficult to get children to find an interest and intellectual satisfaction in the subject.

A number of members took part in the discussion which followed.

A MEETING was held on Saturday, November 9, at Bedford College. There was an attendance of 65, with Mr. Katz in the Chair.

Mr. G. T. Clark read a paper on the teaching of Geometry in Central Schools. The preliminary course should be a discovery course: as it proceeds a large number of riders should be done. Failure to solve riders he considered due to temperamental rather than to mental reasons.

A brisk discussion followed in which Messrs. Durell, Inman, Kearney, Barnard, Atkinson, Katz, Boon, Dr. White and Miss Hornby took part.

F. C. BOON (Hon. Sec.).

MATHEMATICAL ASSOCIATION.

(YORKSHIRE BRANCH.)

THE Autumn Meeting of the Yorkshire Branch of the Mathematical Association was held at Leeds University on October 26, 1929, when twelve new members were elected.

Mr. A. B. Oldfield, A.K.C., of the Secondary School, Pudsey, was elected President in succession to Mr. W. Peaker, B.Sc. Mr. A. Montagnon, M.A., of Leeds Grammar School, was elected Secretary, and Messrs. W. Newbold, M.A.; H. Clayborn, B.Sc., were elected members of the Committee.

After the balance sheet (presented by Miss L. Watson, M.A., Treasurer) had been adopted, Mr. J. C. Gregory, B.Sc., F.I.C., of Leeds University, gave a paper on "Mathematics and Thought." This was followed by a paper from Miss E. Appleyard, of Thoresby High School, entitled "The Teaching of Arithmetic to Young Children."

A. B. OLDFIELD (Hon. Sec.).

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Chapters I-XIV of this edition have been reproduced with some corrections and additional references. Chapters XV (The General Theory of Orbits) and XVI (Integration by Series) have been completely rewritten in order to present the subject as it has been developed by the researches of the last eleven years.

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